

**A Thesis Submitted for the Degree of PhD at the University of Warwick**

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An Orbifold Approach to

Black and White Crystallographic Groups

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(i)

Summary:

Given a crystallographic space group  $G$ , Bonahon and Siebenmann show in [B + S] that it can be thought of as the fundamental group of a closed 3-orbifold  $Q$  which, because in most cases  $G$  preserves some direction  $V$  in  $\mathbb{R}^3$ , usually admits an  $S^1$ -fibration over a 2-orbifold  $B$ : we write

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B.$$

Readers familiar with the definitions and notations for orbifolds and crystallographic groups may wish to omit §0, where these ideas are introduced, dipping back into it only when necessary.

Using these methods, Bonahon and Siebenmann give a new and entirely topological classification of the crystallographic groups, depicting  $Q \rightarrow B$  by a convenient diagram; their methods are described briefly in §1. However, they make no attempt to link this new classification with the existing one i.e. to determine which orbifolds correspond to which crystallographic groups; this is done here, for the first time, in Table 4.

Given an index two subgroup  $G'$  of a crystallographic group  $G$ , the pairs  $(G, G')$ , classified up to affine homeomorphism of  $\mathbb{R}^3$ , are known as black and white groups. In terms of orbifolds they correspond to fibred double covers  $Q' \rightarrow B'$  of  $Q \rightarrow B$ . Such covers for the local structure of fibred orbifolds are constructed in §2 and summarized in Table 3; §4 and §5 then show how to piece them together to form global covers. In §3 we prove that, whenever there is a direction  $V$  in  $\mathbb{R}^3$  which gives a unique fibration, the obvious notion of equivalence for two such covers corresponds exactly to the standard definition of equivalence for black and white groups.

In §6 we deal with those groups whose corresponding orbifolds cannot be fibred; only the orientable orbifolds in the list given here have been described before (in [Du 1]). Finally, in §7, we demonstrate, by means of examples, how to classify the black and white groups by constructing double covers of orbifolds.

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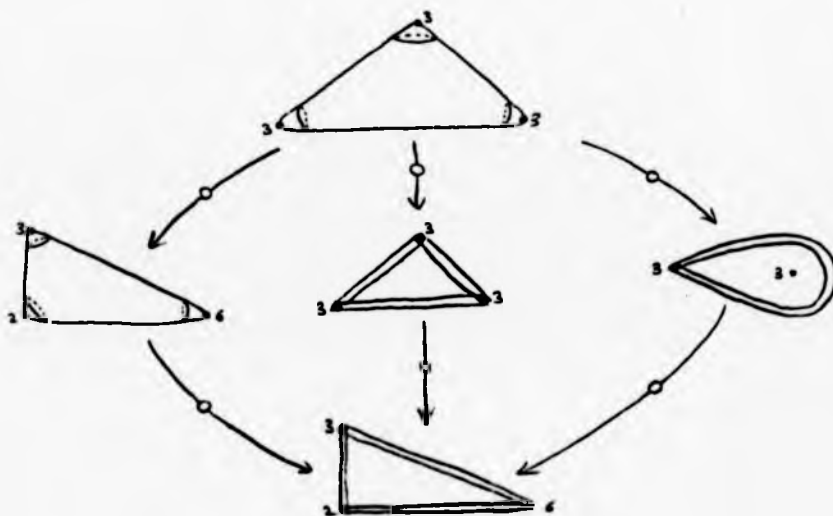
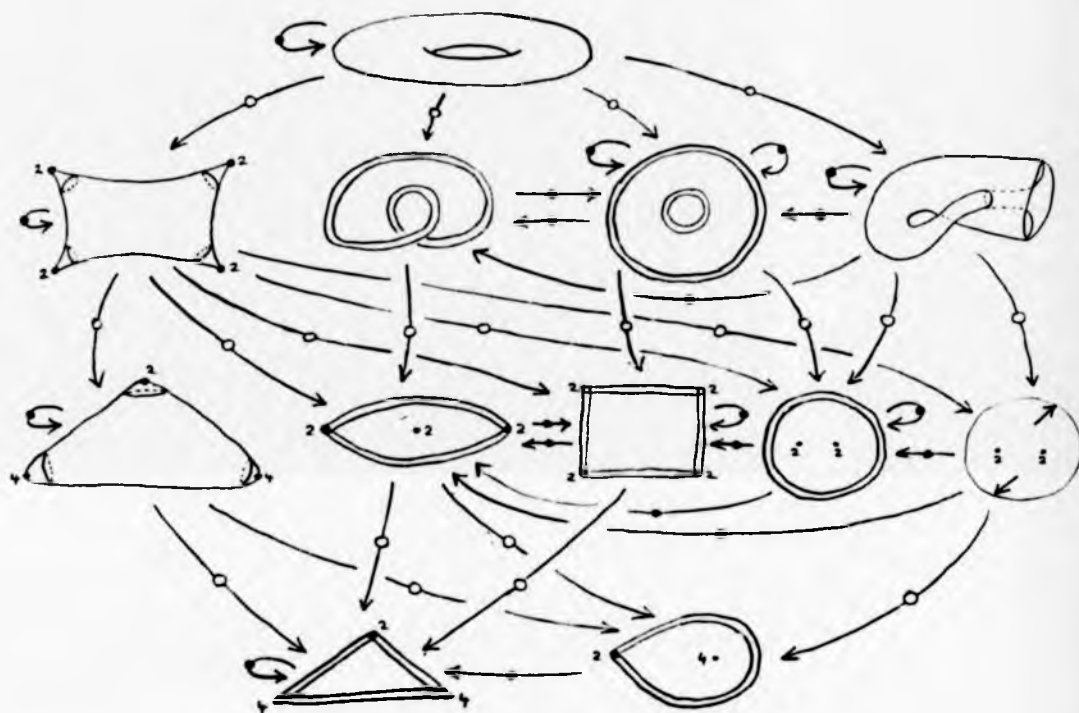
I would also like to thank the S.E.R.C. and the Warwick supervision system for their financial support from 1980-83, and the D.H.S.S. for invaluable aid thereafter.

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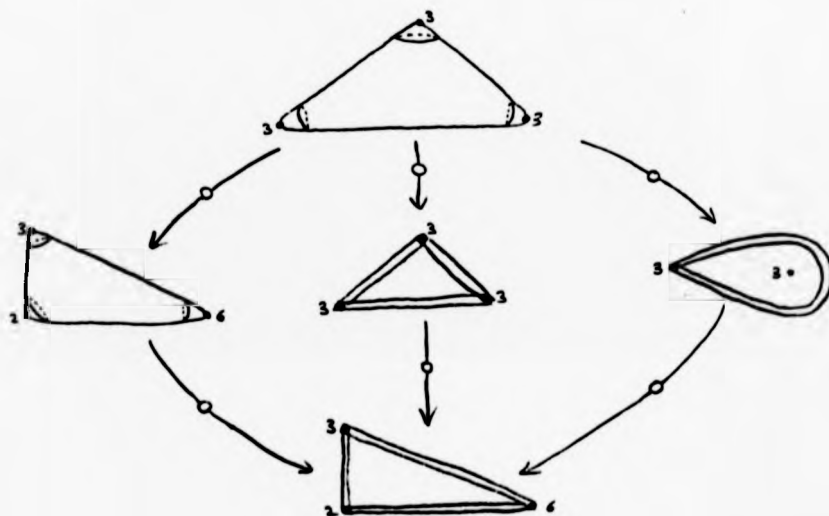
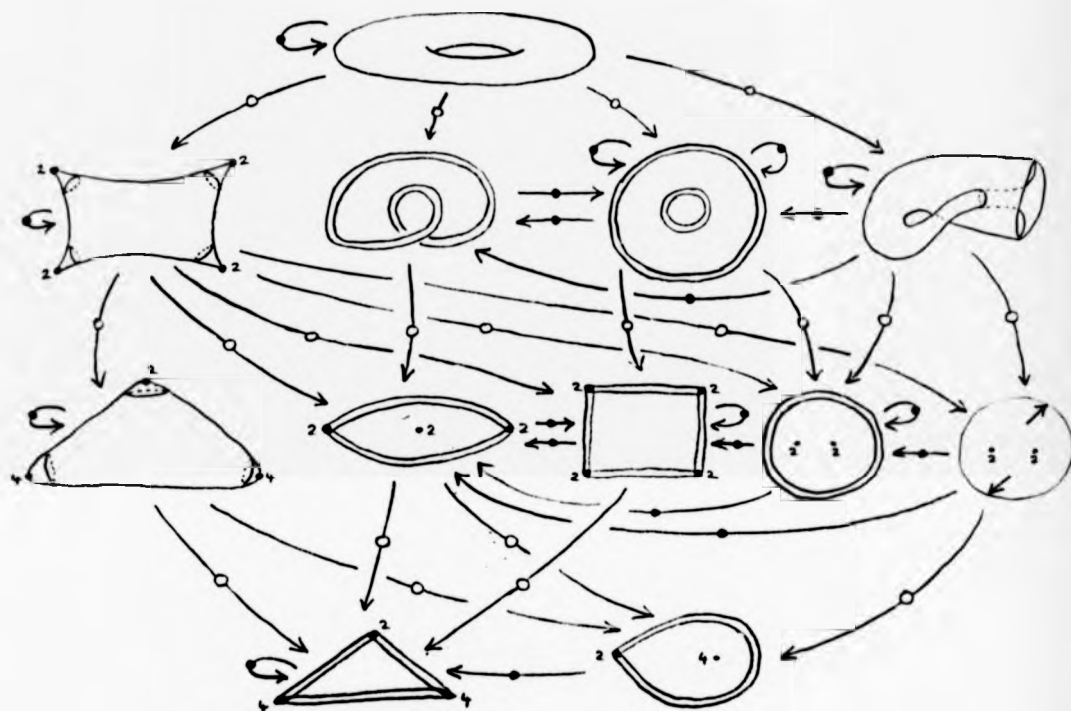
Declaration

I declare that none of the work presented here as my own has been accepted in any previous application for a degree.

## Space Groups



Task 1      2-Dimensional black and white Crystallization  
Spec. Graph:



(iv)

Explanation of Table 1

Table 1 provides a 2-dimensional foretaste of what this thesis aims to do for 3 dimensions. Every 2-dimensional crystallographic space group has a corresponding (unique) closed euclidean 2-orbifold (a 2-crystal - see Table 2). Hence every 2-dimensional black and white group (see Definition 3.0) can be thought of as a double covering of one such 2-crystal by another (or itself) as we have here.

Each arrow in the diagram represents such a group; note that there are two distinct ways by which the annular 2-crystal can cover itself, with one arrow corresponding to each. Arrows denoting Type III groups are marked by a 'o' and those denoting Type IV groups by a '•' (see Definitions 3.1.3 and 3.1.4).



§0. Introduction.

Mathematical crystallography - the study of discrete groups of euclidean isometries - was initiated in the last century, in particular by the work of Barlow [Bar], Fedorov [Fed] and Schönflies [Scho], who all contributed to the classification of the 3-dimensional space groups. Although the motivation was a desire to understand natural crystalline substances, the subject has since acquired a life of its own.

Denoting by  $\mathbb{E}^n$  the euclidean isometry group of  $\mathbb{R}^n$  (sometimes called the *galilean group*) we have the following split short exact sequence:

$$(0.1) \quad 0 \rightarrow \mathbb{R}_0^n \rightarrow \mathbb{E}^n \rightarrow O(n) \rightarrow 1, .$$

where  $\mathbb{R}_0^n \leq \mathbb{E}^n$  is the normal abelian subgroup of translations in  $\mathbb{E}^n$  and  $O(n)$  is the  $n$ -dimensional orthogonal group. To save looking at trivial examples we demand that crystallographic groups be of full dimension.

(0.2) Definition: An  $n$ -dimensional crystallographic space group (or Fedorov group)  $G$  is a discrete subgroup of  $\mathbb{E}^n$  such that  $\mathbb{R}^n/G$  is compact. //

Writing  $G_\infty$  for the image of  $G$  in  $O(n)$  and  $G_0 = G \cap \mathbb{R}_0^n$ , (0.1) restricts to:

$$(0.2.1) \quad 0 \rightarrow G_0 \rightarrow G \rightarrow G_\infty \rightarrow 1 ,$$

which need no longer be split.  $G_\infty = G/G_0$  is called the *holonomy group* or *point group* of  $G$  and  $G_0$  is known as the *translation subgroup* or *lattice*, the last name also being used for the image by  $G_0$  of the origin in  $\mathbb{R}^3$ .

(0.2.2) Theorem [Th:4.3.1.]: For a crystallographic space group  $G$ ,

$$G_\infty = G/G_0 \text{ is finite and } G_0 = G \cap \mathbb{R}_0^n \text{ is of rank } n .$$

□

This theorem, first proved by Bieberbach [Bie 1], shows that  $G$  is of full dimension as required. Elements of  $G$  are often written in the form  $(h, \phi) \in G_0 \times G_\infty$ ; they combine by the rules:

$$(0.2.3) \quad \begin{aligned} (h_1, \phi_1)(h_2, \phi_2) &= (h_1 + \phi_1 h_2, \phi_1 \phi_2) \\ (h, \phi)^{-1} &= (-\phi^{-1} h, \phi^{-1}) . \end{aligned}$$

The study of crystallographic groups has been dominated by the practical needs of chemists and physicists and the above notation lends itself to their requirements. There are vast tables, notably the "International Tables for X-ray Crystallography", listing, amongst much else, the 3-dimensional space groups by means of their generators, in a variation of the above form known as the *Seitz space-group representation*

[ITC, B + C]. Any group can then be realized by seeing how and where it transforms a suitable motif placed at the origin in  $\mathbb{R}^3$ .

(0.2.4) Example: The tetragonal group  $P4/nbm$  (no. 125 - see Table 4).



The bottom left hand corner represents the origin of  $\mathbb{R}^3$ , with axes as shown. Lattice points (i.e. those which can be reached by a pure translation in the group) are marked as solid dots. We can also rotate through

$180^\circ$  about the  $O_x$  and  $O_y$  axes, reflect in a vertical plane lying between them, and rotate through  $180^\circ$  about another line which misses the origin. We imagine the diagrams as being reproduced infinitely in all directions. //

Instead of studying the group-action in this way, leading to these cumbersome diagrams, an alternative approach is to consider the compact quotient required by Definition (0.2), which necessarily carries all the information about the group, albeit in a coded form. This is exactly what Bonahon and Siebenmann have done in [B + S], using the fact that if  $G$  is a crystallographic space group then  $Q = \mathbb{R}^3/G$  is a 3-orbifold.

(0.3) Definition: Let  $|Q|$  be a paracompact Hausdorff space. An  $n$ -orbifold atlas on  $|Q|$  is a covering of  $|Q|$  by a collection of open

sets  $\{U_i\}$ , closed under finite intersections, such that each  $U_i$  has an associated finite group  $\Gamma_i$ , an action of  $\Gamma_i$  on an open subset  $\tilde{U}_i$  of  $\mathbb{R}^n$ , and a homeomorphism  $\phi_i: U_i \rightarrow \tilde{U}_i/\Gamma_i$ . Whenever  $U_i \subset U_j$  there is required to be a monomorphism  $f_{ij}: \Gamma_i \rightarrow \Gamma_j$  and an embedding  $\tilde{\phi}_{ij}: \tilde{U}_i \rightarrow \tilde{U}_j$ , equivariant with respect to  $f_{ij}$ , such that the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{U}_i & \xrightarrow{\tilde{\phi}_{ij}} & \tilde{U}_j \\
 \downarrow & & \downarrow \\
 \tilde{U}_i/\Gamma_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j/f_{ij}\Gamma_i \\
 \uparrow & & \uparrow \\
 U_i & \xrightarrow{\quad} & U_j
 \end{array}$$

An *n-orbifold*  $Q$  is a pair consisting of a space  $|Q|$  and a maximal  $n$ -orbifold atlas.  $|Q|$  is called the *underlying space* of  $Q$ . //

(0.3.1) Example: Although any manifold is an orbifold (with trivial group-actions on all charts), the underlying space of an orbifold need not be a manifold, e.g. the cone on  $\mathbb{R}P^2$ , which can be thought of as the quotient of the 3-ball by an involution. As an orbifold, all its local groups  $\Gamma_x$  are trivial except at the cone point, for which  $\Gamma_x \cong \mathbb{Z}_2$ . //

(0.3.2) Definition: A group  $G$  acts *properly discontinuously* on a locally compact space  $X$  if for every compact  $K \subset X$  there are only finitely many  $g \in G$  such that  $gK \cap K \neq \emptyset$  . //

In particular, crystallographic groups act properly discontinuously.

(0.3.3) Proposition [Th:5.2.4]: If  $M$  is a manifold and  $G$  is a group acting properly discontinuously on  $M$  then  $Q = M/G$  is an orbifold.  $\square$

Orbifolds were introduced by Satake [Sat 1,2], who called them "V-manifolds"; they surfaced again in the work of Thurston, to whom they owe their re-christening. They are natural objects for dealing with crystallographic groups from the topological point of view. Because potted versions of their basics abound, this summary has been kept to the minimum for what is needed here; lengthier treatments are available in [Th] and [Sco], whilst [B + S] and [Du 1] cover the same opening ground as here, together with much else that is of relevance. Most of the crystallography can be found in [Schw 1].

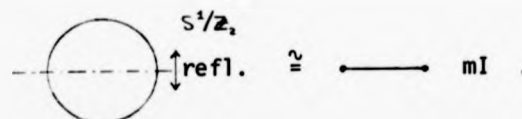
(0.4) Definition: By taking sufficiently small neighbourhoods about a point  $x$  in an orbifold  $Q$ , it is seen to have associated with it a unique minimal finite group  $\Gamma_x$  called the *local group at  $x$* . The *singular set*  $\Sigma_Q$  is defined as:

$$\Sigma_Q = \{x \in Q : \Gamma_x \neq \{id.\}\} ,$$

and points with non-trivial local group  $\Gamma_x$  are said to be *singular* of order  $|\Gamma_x|$ . Points of  $Q - \Sigma_Q$  are called *non-singular* or *regular*. //

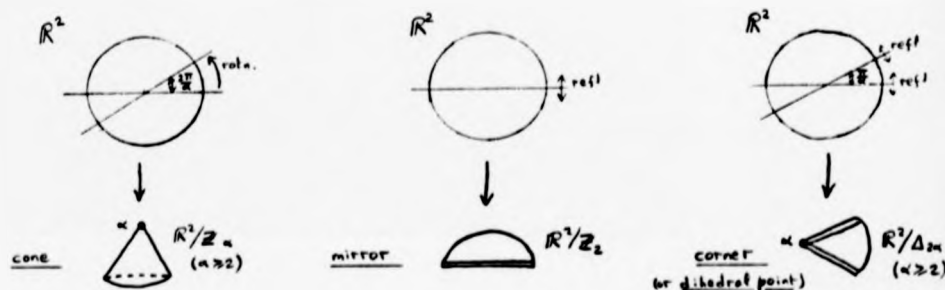
We shall deal solely with differentiable orbifolds, for which it is required in (0.3) that  $\phi_{ij}$  and the group-actions be smooth. It follows [Th:5.4] that if  $x \in Q^n$  then  $\Gamma_x \leq O(n)$  (the  $n$ -dimensional orthogonal group), and from a knowledge of how this group acts we can determine the local structure of (differentiable) orbifolds in each dimension.

(0.4.1) The closed connected 1-orbifolds are the circle  $S^1$  and the circle factored out by  $\mathbb{Z}_2$  acting as a reflection (with two fixed points)



The latter object, which we call  $mI$ , has underlying space a closed interval.

(0.4.2) Since all finite subgroups of  $O(2)$  are cyclic of order  $\alpha$  ( $\mathbb{Z}_\alpha$ ) or dihedral of order  $2\alpha$  ( $\Delta_{2\alpha}$ ), locally all 2-orbifolds must look like:



together with non-singular points modelled on  $\mathbb{R}^2$ . Thus the singular set lies as a 1-dimensional subset of mirrors and corners (which we shall often take the liberty of referring to as  $\partial|Q|$ ) together with a number of isolated cone points. We can therefore portray a 2-orbifold  $Q$  as a 2-manifold  $|Q|$  (perhaps with boundary) with the corners on  $\partial|Q|$  and cone points in  $\text{int}|Q|$  marked and labelled with the order of the maximal cyclic subgroup of their local group, so cone points modelled on  $\mathbb{R}^2/\mathbb{Z}_\alpha$  are denoted by ' $\alpha$ ', as are corners modelled on  $\mathbb{R}^2/\Delta_{2\alpha}$ . A connected component of  $\partial|Q|$  consisting entirely of mirror and corner points is known as a *mirror-cycle*.

(0.4.3) Example:



This is a 2-orbifold with two mirror-cycles.

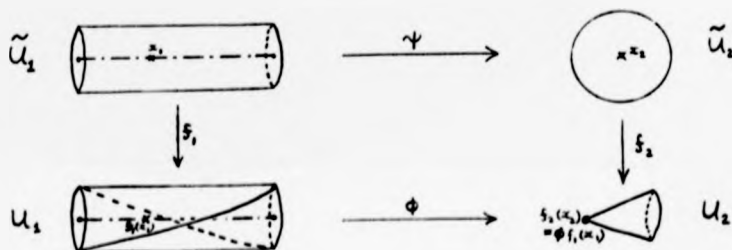
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The singular set for a 3-orbifold is described in [Th:5.6]; it can be drawn as a 1-complex labelled exactly as for the 2-dimensional case except that, to avoid clutter, we suppress any '2's that would appear. In general for an  $n$ -orbifold it is a closed set with empty interior [Th:5.2.5] and so has codimension one or more in  $Q^n$ . Orbifolds are *compact* and *connected* exactly when their underlying spaces are. The notion of an orbifold being (*locally*) *2-sided* and having a (*regular*) *neighbourhood* also carry over from the manifold case.

(0.4.4) Definition: A *geometry*  $(X^n, G)$  is a simply-connected  $n$ -dimensional Riemannian manifold  $X^n$ , with transitive isometry group  $G$ , which covers some compact manifold (with covering translations in the group). //

If we use charts  $\tilde{U}_i \subseteq X^n$  for some geometry  $(X^n, G)$  and require that the group actions preserve the metric, the resulting orbifold is said to be *geometric with model*  $(X, G)$ . When  $(X, G) \cong (\mathbb{R}^n, \mathbb{E}^n)$  i.e. euclidean  $n$ -space (abbreviated to just  $\mathbb{E}^n$ ), this corresponds to the conventional euclidean crystallographic groups with which we shall be concerned throughout. Bonahon and Siebenmann refer to the quotient by such a group as an *n-crystal*. Although the name is somewhat inapt, since the object corresponding to the crystal in nature is really the universal covering space with its accompanying group action, we adopt this terminology as a convenient abbreviation of "closed euclidean  $n$ -orbifold". Although we shall not do so, notice that had we taken any other geometry e.g.  $\mathbb{H}^n$ ,  $\mathbb{S}^n$ , Nil [Sco], this would still give a perfectly sensible notion of a crystallographic group - see [Du 2].

(0.4.5) Definition: A *map* of orbifolds  $\phi: Q_1 \rightarrow Q_2$  is any map between the underlying spaces which makes the atlases compatible:





i.e. if  $f_1: \tilde{U}_1 \rightarrow U_1$  and  $f_2: \tilde{U}_2 \rightarrow U_2$  are charts for  $Q_1$  and  $Q_2$  respectively then for all  $x_1 \in \tilde{U}_1$  and  $x_2 \in \tilde{U}_2$  such that

$$f_2(x_2) = \phi f_1(x_1) ,$$

there exists a well-defined map  $\psi: \tilde{V}_1 \rightarrow \tilde{V}_2$  from an open neighbourhood of  $x_1$  to an open neighbourhood of  $x_2$  such that

$$f_2 \psi(x) = \phi f_1(x) \quad \text{for all } x \in \tilde{V}_1 .$$

In particular, an *isomorphism* of orbifolds is a map which restricts to a homeomorphism between the underlying spaces. //

(0.4.6) Definition: An orbifold  $Q$  is *orientable* if it has an atlas  $\{U_i\}$  for which the  $\tilde{U}_i$  are oriented, all the group actions are orientation-preserving, and so are the embeddings  $\phi_{ij}: U_i \rightarrow U_j$  of Definition (0.3). In dimensions 2 and 3 this is equivalent to  $|Q|$  being a closed orientable manifold. //

(0.4.7) Definition:  $F \subset Q$  is called a *closed suborbifold* if  $|F| \subset |Q|$  and, locally, we have  $(U, V) \cong (\tilde{U}, \tilde{V})/\Gamma$ , where  $\Gamma$  is a finite group acting diagonally,  $|V| \subset |U| \subset |Q|$  and  $|V| \subset |F|$ ; i.e. the local groups for  $Q$  transfer naturally to  $F$ . //

If we add to Proposition (0.3.3) the requirement that the action of  $G$  be *free*, i.e. without fixed points, then  $M/G$  is a manifold and  $M \rightarrow M/G$  is a covering map. Moreover, if  $M$  is simply-connected we have  $\pi_1(M/G) \cong G$ .

(0.5) Definition: A projection  $p: \tilde{Q} \rightarrow Q$  between orbifolds is said to be a *covering map* if each  $x \in Q$  has a chart  $U = \tilde{U}/\Gamma$  such that every component of  $p^{-1}(U)$  is isomorphic, respecting the projection, to  $\tilde{U}/\Gamma_i$  for some  $\Gamma_i \leq \Gamma$ . The *degree* of such a covering is the cardinality of the set  $p^{-1}(x)$  for any regular point  $x \in Q$ .

If  $Q = X/G$ , where  $X$  is simply-connected, then the *fundamental group* of  $Q$  is  $\pi_1(Q) \cong G$ . //

(0.5.1) Example: Provided it is done carefully,  $\pi_1(Q)$  can be defined in terms of paths in  $Q$ . Looking at the two 1-orbifolds of (0.4.1), we first have  $S^1 \cong \mathbb{R}^1/\mathbb{Z}$ ; this has  $\pi_1(S^1) \cong \mathbb{Z}$  and the paths are the loops around the circle. Secondly, we have  $m\mathbb{I} \cong \mathbb{R}^1/(\mathbb{Z}_2 * \mathbb{Z}_2)$ , where  $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b : a^2 = b^2 = \text{id} \rangle \cong \mathbb{Z} \times \mathbb{Z}_2$ , the infinite dihedral group.

This group acts by (evenly spaced) reflections along the length of  $\mathbb{R}$ .



Choosing a suitable base point in  $m\mathbb{I}$  we see that its group of covering

translations  $\pi_1(M)$  is generated by two "reflections"  $a$  and  $b$  in the mirrors at either end. //

(0.5.2) Definition: Given a compact orbifold  $Q$  with a cell decomposition whose interior points have constant local group, the *Euler characteristic* is defined to be

$$\chi(Q) = \sum_{c_i} (-1)^{\dim c_i} \frac{1}{|r(c_i)|} ,$$

where  $c_i$  ranges over the cells and  $|r(c_i)|$  is the order of each group. //

(0.5.3) Proposition [Th:5.5.2]: If  $\tilde{Q} \rightarrow Q$  is an orbifold covering of degree  $k$ , then

$$\chi(\tilde{Q}) = k\chi(Q) .$$

□

Thus, if a geometric orbifold is to be *euclidean* i.e. have universal cover  $\mathbb{R}^n$ , then it must have zero Euler characteristic. In particular, for a closed 2-orbifold  $B$  it can be shown [Th:5.5] that

$$(0.5.4) \quad \chi(B) = \chi(|B|) - \sum_{i=1}^m (1 - 1/q_i) - \frac{1}{2} \sum_{j=1}^n (1 - 1/r_j) ,$$

where  $\chi(|B|)$  denotes the conventional Euler characteristic of the 2-manifold  $|B|$ , the  $q_i$ 's are the orders of the cone points of  $B$ , and the  $r_j$ 's

are the orders of the corners. Using this it is possible to list all the 2-crystals - the quotients of the 2-dimensional crystallographic space groups. There are exactly 17 of them and they are shown in Table 2; the notation is that of (0.2.1).


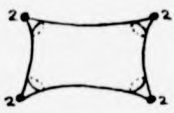


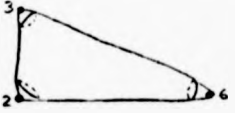











(0.6) Definition: An *orbifold F-fibration*  $Q \xrightarrow{p} B$  is a projection  $p: |Q| \rightarrow |B|$  such that each  $x \in B$  has a chart  $U = \tilde{U}/\Gamma$  such that, for some action of  $\Gamma$  on  $F$ ,  $p^{-1}(U) = (\tilde{U} \times F)/\Gamma$  (where  $\Gamma$  acts diagonally). The product structure should be consistent with  $p$  i.e. the following diagram must commute:

$$\begin{array}{ccc} \tilde{U} \times F & \xrightarrow{\quad / \Gamma \quad} & p^{-1}(U) \subseteq Q \\ \text{projn.} \downarrow & & \downarrow p \\ \text{first} & \text{to} & \\ & \text{factor} & \\ \tilde{U} & \xrightarrow{\quad / \Gamma \quad} & U \subseteq B \end{array}$$

$B$  is called the *base* of the fibration and the *generic fibre* is  $p^{-1}(x)$  for any non-singular point  $x \in B$ ; it may be a quotient of, rather than equal to,  $F$ . For instance, we shall deal with  $S^1$ -fibred orbifolds with generic fibre  $m\mathbb{I}$ . //

(0.6.1) Definition: An *(n-fold) section* of an  $S^1$ -fibration  $Q \xrightarrow{p} B$  is a locally 2-sided closed 2-suborbifold  $F \subset Q$  such that  $p|_F : F \rightarrow B$  is an  $n$ -fold covering map of orbifolds. //

Table 2 The 17 2-Crystals

$G_n$	$B \cong \mathbb{R}^2/G$	
$Z_1 = \{1\}$		$ B  = \text{torus}$
$Z_2$		$ B  = \text{sphere}$
$Z_3$		$ B  = \text{sphere}$
$Z_4$		$ B  = \text{sphere}$
$Z_6$		$ B  = \text{sphere}$
$\Delta_2$	 $ B  = \text{annulus}$	 $ B  = \text{Möbius band}$
$\Delta_4$	 $ B  = \text{disc}$	 $ B  = \text{disc}$
$\Delta_6$	 $ B  = \text{disc}$	 $ B  = \text{disc}$
$\Delta_8$	 $ B  = \text{disc}$	 $ B  = \text{disc}$
$\Delta_{12}$	 $ B  = \text{disc}$	
		 $ B  = \text{Klein bottle}$
		 $ B  = \text{projective plane, } \mathbb{RP}^2$

(0.6.2) Definition: Let  $Q \xrightarrow{p} B$  be an  $S^1$ -fibration which is closed, connected and oriented (so in particular the generic fibre must be  $S^1$  and not  $m\mathbb{I}$ ). Let  $N(x_0)$  be a closed neighbourhood of a non-singular point  $x_0 \in B$  such that  $p^{-1}N(x_0) \cong D^2 \times S^1$ , a regularly-fibred solid torus. There exists an orbifold section of the bundle over the complement of  $N(x_0)$  whose boundary can be written as

$$\partial F_0 = \alpha a + \beta b$$

in  $H_1(\partial p^{-1}N(x_0); \mathbb{Z})$ , where  $a$  and  $b$  are the (suitably oriented) meridian and fibre of  $\partial p^{-1}N(x_0)$ . The *Euler number* is defined to be

$$e_0(p) = -\beta/\alpha ;$$

it is an invariant of  $Q \xrightarrow{p} B$  [Du 1:2.1.3] and represents the obstruction to extending  $F_0$  to a global section.

A (closed, connected) non-orientable  $S^1$ -fibration  $Q \xrightarrow{p} B$  always admits such a section and so in such cases we define  $e_0(p) = 0$ . //

(0.6.3) Proposition: If

$$\begin{array}{ccc} Q' & \xrightarrow{\phi} & Q \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{\bar{\phi}} & B \end{array}$$

is a covering map of closed, connected  $S^1$ -fibred 3-orbifolds sending fibres to fibres with degree  $d$  and projecting to a covering map  $\bar{\phi} : B' \rightarrow B$  of degree  $\delta$  then

$$e_0(Q') = (\delta/d) e_0(Q) .$$

□

The following theorem of Bieberbach [Bie 2] tells us that classifying the 219 3-dimensional crystallographic space groups up to conjugacy by affine homeomorphisms of  $\mathbb{R}^3$  is the same as classifying them up to ordinary group-isomorphism:

(0.6.4) Theorem [Schw 1:3.2]: Any isomorphism  $\theta: G_1 \rightarrow G_2$  between  $n$ -dimensional crystallographic space groups is realized by an affine homeomorphism  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  i.e.

$$\theta(g) = fgf^{-1} \quad \text{for all } g \in G_1.$$

□

In practice it is usual to classify up to conjugation by orientation-preserving affine homeomorphisms, leading to 230 distinct groups; the 11 pairs which differ from one another by an orientation-reversing map are called *amphicheiral pairs*. It follows that equivalence at the orbifold level is by orientation-preserving affine isomorphism or, equivalently by the above theorem, just by orientation-preserving smooth isomorphism. This is exactly how Bonahon and Siebenmann performed the classification in [B + S:§Y], from which come Propositions (0.7.1) to (0.7.6). In what follows,  $G$  is a (euclidean) crystallographic space group.

(0.7) Definition: Given a 1-dimensional linear subspace  $V = \{\lambda \underline{v} : \lambda \in \mathbb{R}\}$  of  $\mathbb{R}^3$  (for some non-zero vector  $\underline{v} \in \mathbb{R}^3$ ), the cosets of the form  $\underline{a} + V$ , where  $\underline{a} \in \mathbb{R}^3$ , determine a (geodesic) fibration of  $\mathbb{R}^3$  which we denote by

$\mathbb{R}^3/V = \{ \underline{a} + V : \underline{a} \in \mathbb{R}^3 \}$  .  $\mathbb{R}^3$  is said to be *fibred in direction V* and a group  $G$  *preserves V* if  $\mathbb{R}^3/V$  is preserved by the action of  $G$  on  $\mathbb{R}^3$  .

Such a fibration which passes through at least two lattice points of a group  $G$  is called a *rational fibration* for  $G$  . //

(0.7.1) Proposition: A geodesic  $S^1$ -fibration of a crystal  $\mathbb{R}^3/G$  lifts to a fibration of  $\mathbb{R}^3$  whose direction  $V$  satisfies:

- (i) it is preserved by  $G_\infty$  .
- (ii)  $G_0$  contains translations in direction  $V$  .

Conversely, such a family in  $\mathbb{R}^3$  yields a geodesic  $S^1$ -fibration of  $\mathbb{R}^3/G$  .  $\square$

(0.7.2) Proposition: If  $G_\infty$  contains a non-trivial rotation or reflection  $g$  which preserves  $V$  then (0.7.1(ii)) holds i.e.  $G_0$  contains translations in direction  $V$  . Moreover,  $G_0$  contains a rank 2 subgroup of translations perpendicular to  $V$  .  $\square$

These two Propositions tell us that, provided  $G_\infty$  contains an element  $g$  as in (0.7.2) (which, for most  $G$  , it does), by fibring  $\mathbb{R}^3$  in the preserved direction  $V$  this projects to a geodesic  $S^1$ -fibration of  $Q = \mathbb{R}^3/G$  , which we write as:

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B .$$



(0.7.3) Proposition: A closed connected  $S^1$ -fibred 3-orbifold  $Q \xrightarrow{p} B$  is smoothly isomorphic to a 3-crystal if and only if  $B$  is euclidean and  $e_0(p) = 0$ .  $\square$

(0.7.4) Proposition: If  $Q \xrightarrow{p} B$  is isomorphic to a crystal  $\mathbb{R}^3/G$  then it is fibred-isomorphic to a geodesically  $S^1$ -fibred crystal.  $\square$

(0.7.5) Proposition: A smooth fibred isomorphism  $f: Q_1 \rightarrow Q_2$  of geodesically  $S^1$ -fibred 3-crystals can be replaced by a fibred affine isomorphism.  $\square$

Using all this machinery, Bonahon and Siebenmann conclude that classifying geodesically  $S^1$ -fibred 3-crystals up to fibred affine isomorphism is the same as classifying all  $S^1$ -fibred 3-crystals up to smooth fibred isomorphism, and that these objects can be recognized by having  $e_0(p) = 0$  and a 2-crystal as base (see Table 2).

Thus it is possible to provide an alternative classification of the crystallographic groups by determining all the  $S^1$ -fibred 3-crystals up to fibred isomorphism. The only obstacles are:

- (i) Some groups  $G$  (36 out of 230) do not preserve a direction in  $\mathbb{R}^3$  (so  $G$  does not satisfy (0.7.2)) and consequently their 3-crystals cannot be fibred. These are dealt with in §6.
- (ii) Some groups  $G$  preserve more than one direction in  $\mathbb{R}^3$  (sometimes infinitely many!), leading to as many as three distinct fibrations of  $\mathbb{R}^3/G$  which need to be collated. The results can be seen in Table 4.

§1. The Local Fibre Types and their Fibred Double Covers.

The first step towards a classification of the  $S^1$ -fibred 3-crystals is to determine their local structure; this is exactly what Bonahon and Siebenmann do in [B + S: §7], where they number the distinct (local) fibre types, both orientable and non-orientable, from 0 to 19. This thesis aims to classify the black and white groups (§3) by constructing fibred double covers of the  $S^1$ -fibred 3-crystals (§§4-5). The first stage in this process is to construct such covers for these local fibre types; this is the purpose of the current chapter. It also affords an opportunity to examine closely these "building blocks" of  $S^1$ -fibred 3-orbifolds and to complement [B + S] by drawing them, so far as is possible.

To make a complete list of the local fibred double covers we take each of the 20 fibre types in turn and factor out by all the possible fibred involutions (fibred double covering translations). The restriction to the fibres and the projection to the base space necessarily have order either one or two, which observation enables us to establish exactly the permitted involutions, since it restricts us to certain actions depending on the nature of the base.

Let  $Q' \xrightarrow{p'} B'$  be a local fibre type with fibred involution  $\rho: Q' \rightarrow Q'$  projecting to  $\rho_B: B' \rightarrow B'$  on the base. If  $B'$  is a cone with angle  $2\pi/\alpha'$  (so  $B' \cong D^2/\mathbb{Z}_{\alpha'}$ ), then  $\rho_B$  can be: the identity map; a reflection; or a rotation through  $\pi/\alpha'$ , all fixing the cone point. If  $B'$  is a corner of angle  $\pi/\alpha'$  (so  $B' \cong D^2/\Delta_{2\alpha'}$ ) then  $\rho_B$  must

be either the identity map or a reflection fixing the dihedral point (this includes the case of a mirror i.e.  $\alpha' = 1$ ).

For the fibres, if  $\rho_f$  is the restriction of  $\rho$  to a generic fibre then when these are  $S^1$ 's we can again have the identity, a reflection, or a rotation. When the fibres are  $mI$ 's, only the identity or a reflection (through the centre of the fibre) are possible.

The results are given in the form of a small table for each fibre type  $Q' \xrightarrow{p'} B'$  listing those  $Q \xrightarrow{p} B$  (with fibre  $f$ ) of which it is the fibred double cover, together with a description of how the covering map acts on the base and fibres of  $Q'$ . Hence we have:

$$\begin{array}{ccc} Q' & \xrightarrow{\phi} & Q \\ \text{fibre } f' \downarrow p' & & \text{fibre } f \downarrow p \\ B' & \xrightarrow{\bar{\phi}} & B \end{array},$$

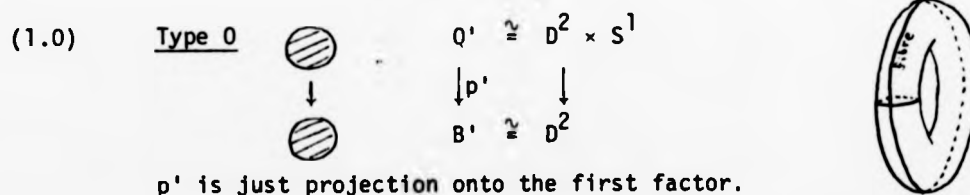
where  $\phi: Q' \rightarrow Q$  is the fibred double covering induced by  $\rho: Q' \rightarrow Q'$  and  $\bar{\phi}: B' \rightarrow B$  is its projection to the base spaces (and so is either a single or a double covering map of 2-orbifolds).

At the end of the chapter these small tables are compiled into the more useful form of Table 3. This expresses the coverings in the opposite way to that in which they were derived i.e. it lists the local fibre types  $Q \xrightarrow{p} B$  followed by those fibre types which are their fibred double covers. It is divided into five sections as follows:






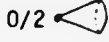

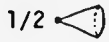








$Q \xrightarrow{p} B$  is double covered by  $Q' \xrightarrow{p'} B'$



fibre $S^1$	fibre $S^1$ and $B'$ double covers $B$ (Table 3.1)
" $mI$	" $mI$ " $B'$ double covers $B$ (Table 3.2)
" $S^1$	" $S^1$ " $\phi: B' \cong B$ (Table 3.3)
" $mI$	" $S^1$ " $\phi: B' \cong B$ (Table 3.4)
" $mI$	" $mI$ " $\phi: B' \cong B$ (Table 3.5)

This breakdown makes life easier for us when we are constructing global double covers of a given fibred 3-crystal (§§4,5), since it enables us to systematically produce covers of a different kind without wading through unnecessary information to do so. The diagrammatic notation for the fibre types is that of  $[B + S]$ . They are not used here in the order in which we describe them, but for explanations the unacquainted reader may turn ahead in this chapter to the relevant fibre types.



$Q'$  is the neighbourhood of a regular  $S^1$ -fibre and so is just a regularly fibred solid torus, a special case of Type 1 which we examine next. Consequently we only summarize the results here.

Action of $\rho$ on Base ( $\rho_B$ )	Action of $\rho$ on central Fibre	B	Q	f	Type of Q+B
id.	rotn.			$S^1$	0
id.	refl.			mI	12
rotn.	id.	2 	0/2 	} $S^1$	1
rotn.	rotn.	2 	1/2 		1
rotn.	refl.	2 			4
refl.	id.				5
refl.	rotn.				6
refl.	refl.				2

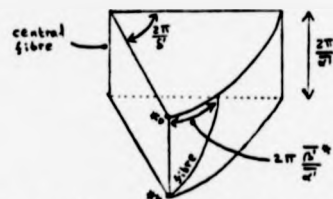
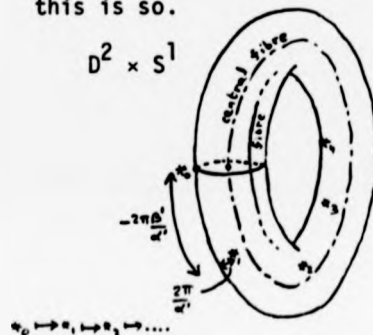
(1.1) Type 1  $\beta'/\alpha'$    
 $\downarrow$   
 $\alpha'$  

$$Q' \cong (D^2 \times S^1) / \mathbb{Z}_{\alpha'}$$

$$p' + \quad +$$

$$B' \cong D^2 / \mathbb{Z}_{\alpha'}$$

The generator of  $\mathbb{Z}_{\alpha'}$  acts by rotating  $D^2$  through  $2\pi/\alpha'$  and  $S^1$  through  $-2\pi\beta'/\alpha'$  (with the usual orientation conventions, so that positive rotations of  $D^2$  are anticlockwise seen from above and positive rotations of  $S^1$  go upwards from  $D^2$ ). We assume that  $-\alpha'/2 < \beta' \leq \alpha'/2$ , since by an orientation-preserving fibred isomorphism we can ensure that this is so.

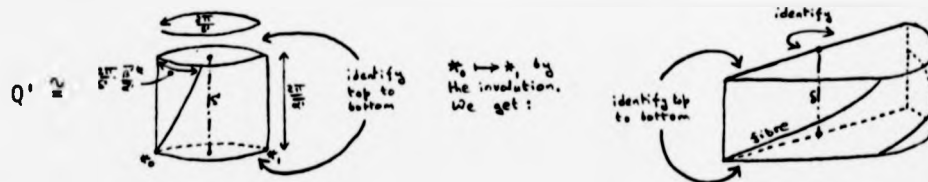


$\delta' = (\alpha', \beta')$  (the H.C.F. of  $\alpha'$  and  $\beta'$ ),  
 $\overline{\alpha'} = \alpha' / \delta'$ ,  $\overline{\beta'} = \beta' / \delta'$   
 and  $\overline{\beta'}^* \overline{\beta'} = 1 \pmod{\overline{\alpha'}}$   
 (so  $\overline{\beta'}^*$  is unique in the range  $(-\overline{\alpha'}/2, \overline{\alpha'}/2]$ ).

On the left is  $D^2 \times S^1$ , indicating the action of the generator of  $\mathbb{Z}_\alpha$ , sending  $*_0$  to  $*_1$  to  $*_2$  etc. on repeated application. On the right is a fundamental domain for this action, looking like a "slice of cake". The fibre is no longer vertical because it has been twisted so that the top and bottom, which are identified by the action of  $\mathbb{Z}_\alpha$ , lie over one another.  $Q'$  is formed by making this identification (by a vertical translation) and then identifying the two vertical sides, by rotating about the central fibre. This yields a solid torus with a twisted fibration. It is exactly the same structure as the twisted fibre neighbourhoods which occur in Seifert manifolds [Sei], except that the central fibre consists of points which are singular of order  $\delta'$  and so, when  $\alpha'$  and  $\beta'$  fail to be co-prime, it is not a manifold. Starting with this description of  $Q' \xrightarrow{p'} B'$ , let us look at various involutions, expressed as  $(\rho_B, \rho_F)$ , and see what occurs.

(1.1.1) (id., rotn.) We deal separately with the cases (a)  $\overline{\alpha'} = \text{even}$ ,  
(b)  $\overline{\alpha'} = \text{odd}$ .

(a)  $\overline{\alpha'} = \text{even}$



We are cutting  $Q'$  vertically in two and "folding up" along the central fibre. If  $\delta, \alpha, \beta$ , etc., are the invariants of the quotient  $Q \xrightarrow{p} B$

then we have

$$\delta = 2\delta' \quad \text{and} \quad \alpha = \alpha'$$

(the latter since the action on the base is the identity). Thus  $\bar{\alpha} = \frac{\alpha'}{2}$ , and equating distances gives

$$\frac{2\pi}{\delta'} \frac{\bar{\beta}^{\tau*}}{\alpha'} = \frac{2\pi}{\delta} \left( \frac{\bar{\beta}^*}{\alpha} + k_1 \right) = \frac{2\pi}{\delta'} \left( \frac{\bar{\beta}^*}{\alpha'} \cdot 2 + k_1 \right) \quad (k_1 \in \mathbb{Z})$$

and hence 
$$\bar{\beta}^* = \bar{\beta}^{\tau*} + \frac{k_1 \alpha'}{2}.$$

We must have 
$$\bar{\beta}^* \bar{\beta} = 1 \pmod{\bar{\alpha}}$$

i.e. 
$$(\bar{\beta}^{\tau*} + k_1 \frac{\alpha'}{2}) \bar{\beta} = k_2 \frac{\alpha'}{2} + 1 \quad (k_2 \in \mathbb{Z}).$$

We know that 
$$\bar{\beta}^{\tau*} \bar{\beta}^{\tau} = k_3 \alpha' + 1 \quad (k_3 \in \mathbb{Z})$$

so, combining equations, we get

$$\bar{\beta}^{\tau*} (\bar{\beta} - \bar{\beta}^{\tau}) = \text{constant} \cdot \frac{\alpha'}{2}.$$

Hence, since  $(\bar{\beta}^{\tau*}, \alpha') = 1$ , we have

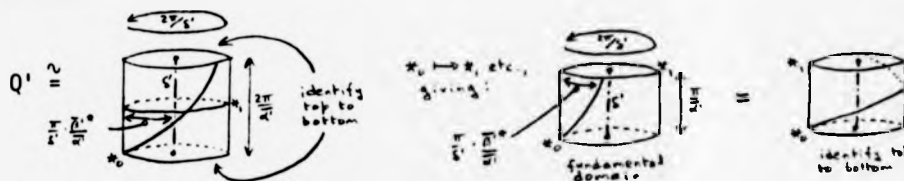
$$\bar{\beta} - \bar{\beta}^{\tau} = m \alpha' / 2 \quad \text{for some } m \in \mathbb{Z}.$$

So 
$$\beta = \delta \bar{\beta} = \delta' (\bar{\beta}^{\tau} + \frac{m \alpha'}{2})$$

$$= 2\delta' \bar{\beta}^{\tau} + \delta' m \alpha' = 2\beta' + m \alpha'.$$

We require  $-\frac{\alpha}{2} < \beta \leq \frac{\alpha}{2}$ , for which there is a unique value of  $m$ .

(b)  $\overline{\alpha^T} = \text{odd}$ .



Now we have  $\delta = \delta'$  and (since  $\alpha = \alpha'$ )  $\bar{\alpha} = \overline{\alpha^T}$ . Further,

$$\frac{2\pi}{\delta} \frac{\bar{\beta}^*}{\bar{\alpha}} = \frac{\pi}{\delta'} (2m+1) + \frac{\pi}{\delta'} \frac{\bar{\beta}^*}{\alpha^T} \quad (m \in \mathbb{Z}),$$

where the  $(2m+1)\pi/\delta'$  rotates the top so that  $*_1$  lies over  $*_0$ .

But

$$\frac{2\pi}{\delta} \frac{\bar{\beta}^*}{\bar{\alpha}} = \frac{2\pi}{\delta'} \frac{\bar{\beta}^*}{\alpha^T},$$

and so we have  $\bar{\beta}^* = \frac{\bar{\beta}^* \alpha^T + (2m+1)\overline{\alpha^T}}{2}$ ,

where  $m \in \mathbb{Z}$  ensures that this lies in the range  $(-\frac{\bar{\alpha}}{2}, \frac{\bar{\alpha}}{2}] = (-\frac{\overline{\alpha^T}}{2}, \frac{\overline{\alpha^T}}{2}]$ .

$\bar{\beta}$  satisfies  $\bar{\beta} \bar{\beta}^* = 1 \pmod{\bar{\alpha}}$ ,

so  $\frac{\bar{\beta}}{2} (\bar{\beta}^* \alpha^T + (2m+1)\overline{\alpha^T}) = k_1 \bar{\alpha} + 1 \quad (k_1 \in \mathbb{Z})$ .

We know that  $\bar{\beta}^* \bar{\beta}^* \alpha^T = k_2 \overline{\alpha^T} + 1 \quad (k_2 \in \mathbb{Z})$

so we get  $\bar{\beta}^* (\bar{\beta} - 2\bar{\beta}^*) = \text{constant} \cdot \overline{\alpha^T}$ ,

which, since  $(\overline{\alpha^T}, \bar{\beta}^*) = 1$ , gives

$$\bar{\beta} - 2\bar{\beta}^* = n \overline{\alpha^T} \quad \text{i.e.} \quad \bar{\beta} = 2\bar{\beta}^* + n \overline{\alpha^T} \quad (n \in \mathbb{Z}).$$



Hence  $\beta = \delta \bar{\beta} = \delta' (2\bar{\beta}' + n\bar{\alpha}') = 2\beta' + n\alpha'$ ,

for which there is a unique  $n \in \mathbb{Z}$  giving  $\beta$  in the range  $(-\frac{\alpha'}{2}, \frac{\alpha'}{2}] = (-\frac{\alpha'}{2}, \frac{\alpha'}{2}]$ .

Thus both cases give the same answer and we have the combined result that:

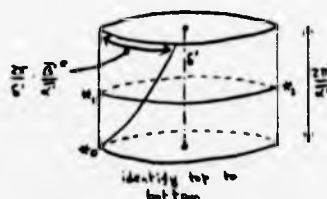
$$\frac{\beta'}{\alpha'} \xrightarrow{(id, rotn.)} \frac{2\beta' + n\alpha'}{\alpha'} \quad \left( \text{where } -\frac{\alpha'}{2} < 2\beta' + n\alpha' \leq \frac{\alpha'}{2} \right)$$

(1.1.2) (rotn., id.) These are best dealt with together.

(1.1.3) (rotn., rotn.) Using the same notation as before, because the action on the base is rotation we immediately have

$$\alpha = 2\alpha'$$

When  $\bar{\alpha}'$  is even we have the following situation:



The involution sends  $*_0$  to  $*_1$  or  $*_2$  and everything else accordingly. Hence

$$\alpha = 2\alpha', \quad \delta = \delta', \quad \bar{\alpha} = 2\bar{\alpha}'$$

and so  $\frac{k_1 \pi}{\delta'} + \frac{\pi}{\delta} \frac{\bar{\beta}^*}{\bar{\alpha}'} = \frac{2\pi}{\delta} \frac{\bar{\beta}^*}{\bar{\alpha}} = \frac{2\pi}{\delta'} \frac{\bar{\beta}^*}{2\bar{\alpha}'}$  (some  $k_1 \in \mathbb{Z}$ )

and therefore  $\bar{\beta}^* = \bar{\beta}'^* + k_1 \bar{\alpha}'$

$\bar{\beta}$  satisfies  $\bar{\beta} \bar{\beta}^* \equiv 1 \pmod{\bar{\alpha}}$

i.e.  $\bar{\beta} (\bar{\beta}'^* + k_1 \bar{\alpha}') = k_2 \cdot 2\bar{\alpha}' + 1$  (some  $k_2 \in \mathbb{Z}$ )

Moreover,  $\bar{\beta}^T \bar{\beta}^{T*} = k_3 \bar{\alpha}^T + 1$  (some  $k_3 \in \mathbb{Z}$ ),

and so  $\bar{\beta}^{T*}(\bar{\beta} - \bar{\beta}^T) = \text{constant} \cdot \bar{\alpha}^T$ .

Since  $(\bar{\alpha}^T, \bar{\beta}^{T*}) = 1$  we get  $\bar{\beta} = \bar{\beta}^T + m \bar{\alpha}^T$  ( $m \in \mathbb{Z}$ )

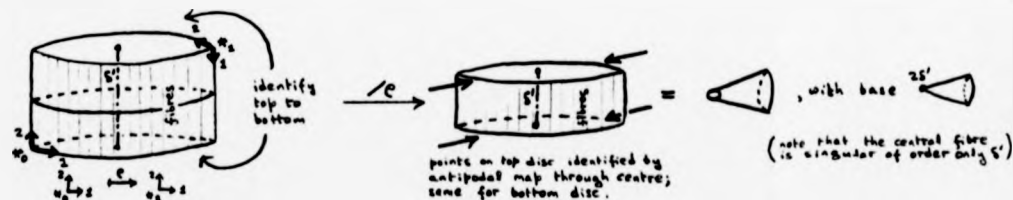
i.e.  $\beta = \beta' + m \alpha'$ .

Since we require  $-\frac{\alpha'}{2} < \beta \leq \frac{\alpha'}{2}$  i.e.  $-\alpha' < \beta \leq \alpha'$ , there are exactly two solutions for  $\beta$  in the required range, namely  $\beta = \beta'$  and either  $\beta = \beta' + \alpha'$  or  $\beta = \beta' - \alpha'$  (but not both).

When  $\alpha'$  is odd the set-up is slightly different, with  $\alpha = 2\alpha'$  and  $\delta = 2\delta'$ , but careful analysis yields the same result. To summarize, we have:



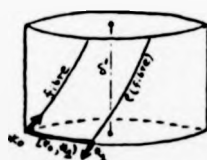
(1.1.4) (rotn.,refl.) When  $\bar{\beta}^{T*} = 0$  i.e. the fibres are vertical, we have:



The boundary of  $Q \xrightarrow{p} B$  is a Klein bottle fibred by orientation-preserving circles; the central fibre is an  $m1$ -fibre.  $Q$  has two non-manifold points, at either end of this fibre; a neighbourhood of such a point has underlying space the cone on  $\mathbb{RP}^2$ .

(1.1.5) Lemma: If in (1.1.4) we have  $\overline{B}^T \neq 0$  i.e. the fibres are not vertical (and note that they cannot be horizontal) then a fibred involution  $\rho$  of the form  $(\rho_B, \rho_f) = (\text{rotn.}, \text{refl.})$  is impossible.

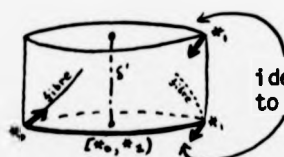
Proof: Let  $Q' \xrightarrow{P'} B'$  be of slope  $\beta'/\alpha' \neq 0$  (so  $\overline{B}^T \neq 0$ ). Choose any fibre in  $\partial Q'$ , together with the distinct fibre to which it is sent by  $\rho$  (distinct because  $\rho$  acts as rotation on the base and so has no fixed points other than the cone point itself). Since  $\rho$  reverses the sense of the fibres, the following set-up occurs somewhere on  $\partial p'^{-1}N(x_0)$ :



(identify top to bottom)

$*_0 \xrightarrow{\rho} *_1$ , and the fibres are sent to one another by  $\rho$  as indicated by the arrows.  $*_0$  and  $*_1$  can be chosen to lie on a geodesic  $[*_0, *_1]$  which is horizontal (i.e. part-bounds a meridian disc with geodesic boundary) and which meets each fibre

exactly once. But since  $*_0 \xrightarrow{\rho} *_1 \xrightarrow{\rho} *_0$ , and  $\rho_B$  is a rotation (so  $\rho([*_0, *_1]) \cap [*_0, *_1] = \emptyset$ ), the above picture must look like:



identify top to bottom

i.e.  $*_0$  lies opposite  $*_1$ .

Since  $[*_0, *_1]$  meets each fibre exactly once, it is clear that the fibres cannot be twisted i.e. they are vertical.  $\square$

To summarize, the only case that occurs is:



$(\text{rotn.}, \text{refl.})$



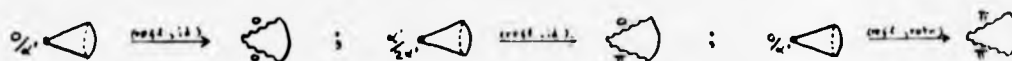
(with base  $2\alpha'$ ).

(1.1.6) (refl.,id.)

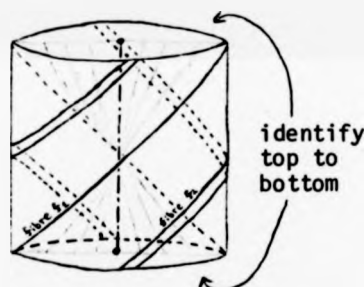
(1.1.7) (refl.,rotn.)

Again, the two cases are best dealt with jointly.

Here  $\rho_f$  refers to the action of  $\rho$  on the central fibre. We certainly have the following three cases:



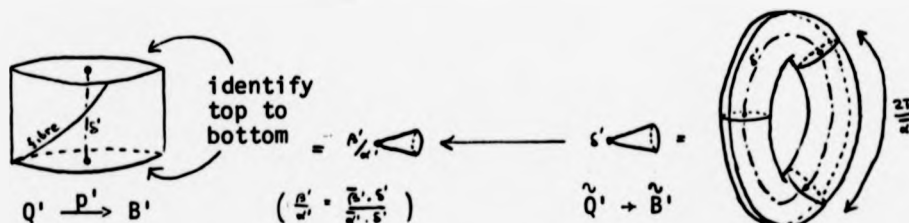
The second of these merits a closer look.



The involution corresponds to a reflection in the shaded "helix" (which is really a Möbius band, since the top is identified to the bottom). This gives straightforward reflections across fibre  $f_1$  (corresponding to  $\frac{0}{2}$ ) and reflections combined with a  $\pi$ -rotation (in the fibre direction) across fibre  $f_2$  (corresponding to  $\frac{\pi}{2}$ ).

(1.1.8) Lemma: The above three cases are the only ones that occur.

Proof: Suppose we have  $\frac{\beta'}{\alpha'} \neq 0, \frac{1}{2}$  (in lowest terms); thus our fibration has an  $\alpha'$ -fold cover  $\tilde{Q}' \rightarrow \tilde{B}'$  as follows:



Any fibre in  $\partial \bar{Q}^T$  lifts to  $\bar{\alpha}^T$  vertical copies of itself in  $\partial \bar{Q}'$ ; note that  $\bar{\alpha}^T \geq 3$ . The fibred involution on  $Q' \xrightarrow{P'} B'$ , which sends exactly two fibres to themselves (not necessarily fixing any points, though), must lift to a fibred involution on  $\tilde{Q}' \rightarrow \tilde{B}'$  which does the same. But, because  $\bar{\alpha}^T \geq 3$ , this cannot be so, for such a map of  $\tilde{Q}'$  to itself cannot be equivariant with respect to the covering of  $Q' \xrightarrow{P'} B'$  as it is required to be.

Finally, observe that (refl., rotn.) cannot occur on  $\frac{\alpha'}{2\alpha'}$  (remember that the rotation refers to the central fibre here).  $\square$

We therefore have only the following three cases:



(1.1.9) The remaining cases when  $Q' \xrightarrow{P'} B'$  is of Type 1 can be dealt with similarly, giving us the following complete list:

$Q'$	Action of $\rho$ on $\text{Base}(\rho_B)$	Action of $\rho$ on Central Fibre	$B$	$Q$	$f$	Type of $Q \xrightarrow{P} B$
	id.	rotn.			$S^1$	1
	id.	refl.			} mI	13
	id.	refl.				14
	{ rotn. rotn. }	{ id. refl. }			} S^1	1
	rotn.	refl.				4
	refl.	id.			} S^1	7
	refl.	id.				9
	refl.	rotn.				8
	refl.	refl.				3

(1.2) Type 2:



$$Q' \cong (D^2 \times S^1) / \mathbb{Z}_2$$

$$p' \downarrow$$

$$B' = D^2 / \mathbb{Z}_2$$



$\mathbb{Z}_2$  acts by reflection on both  $D^2$  and  $S^1$ , so  $Q'$  is orientable.  $\Sigma_{Q'}$  lies as two parallel horizontal lines bounding a band of  $m$ -fibres.

We can perform identity and reflection operations on the base and either the identity or reflection on the  $m$ -fibres, giving the following complete list:

Action of $\rho$ on Base ( $\rho_{B'}$ )	Action of $\rho$ on $m$ -fibres	Action of $\rho$ on $S^1$ -fibres	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	id.	refl. (vert. plane)			} $m$	15
id.	refl.	refl. (hor. plane)				16
id.	refl.	rotn.				2
refl.	id.	id.			} $S^1$	10
refl.	id.	refl. (vert. plane)				3
refl.	refl.	rotn.				11
refl.	refl.	refl. (hor. plane)				3

(1.3) Type 3:



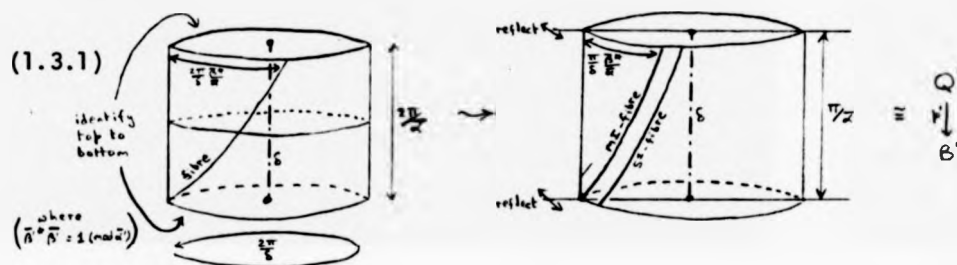
$$Q' \cong (D^2 \times S^1) / \Delta_{2\alpha'}$$

$p' \downarrow$

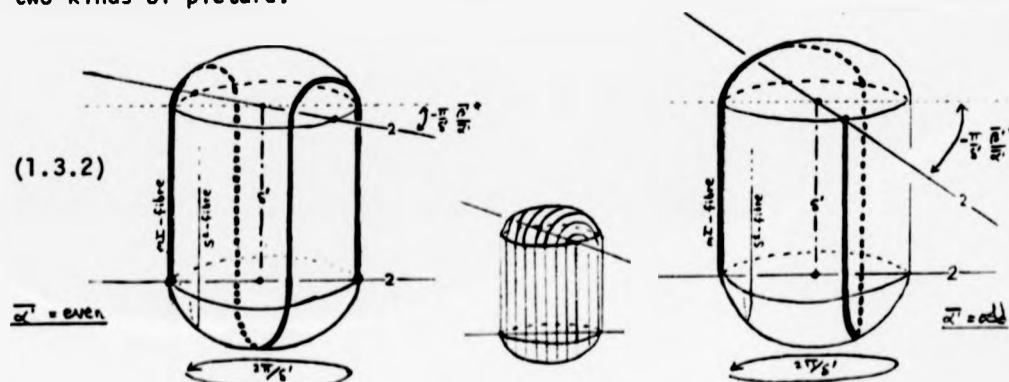
$$B' \cong D^2 / \Delta_{2\alpha'}$$

$\Delta_{2\alpha'}$  is the dihedral group of order  $2\alpha'$ ; we assume that  $\alpha' \geq 2$  since  $\alpha' = 1$  is just Type 2. The generator of  $\mathbb{Z}_{\alpha'} \leq \Delta_{2\alpha'}$  rotates  $D^2$  through

$\frac{2\pi}{\alpha}$  and  $S^1$  through  $-2\pi \frac{\beta'}{\alpha}$  (i.e. just as for a Type 1 fibration with slope  $\frac{\beta'}{\alpha}$ ), whilst each element of  $\Delta_{2\alpha}, -\mathbb{Z}_\alpha$ , acts by reflection on  $D^2$  and  $S^1$ . Thus  $Q$  is the quotient of the aforesaid Type 1 fibration by a fibred involution:

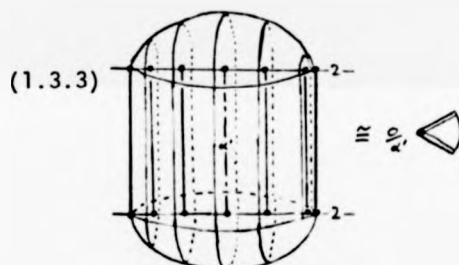


On the left is the Type 1 fibration and on the right is its fibred quotient by an element of  $\Delta_{2\alpha}, -\mathbb{Z}_\alpha$ . The top and bottom of the figure are each identified to themselves by reflection across the diameter of the disc as shown. The singular set  $\Sigma_Q$ , therefore consists of two horizontal lines which are singular of order 2 (being the fixed points of the reflections) together with, when  $\delta' > 1$ , the vertical central mI-fibre, which is singular of order  $\delta'$ . These objects combine to form a letter "H" (on its side). By twisting the two horizontal parts about the axis formed by the vertical spur the fibres can be made to lie vertically, giving one of two kinds of picture.



The distinction is that when  $\bar{\alpha}^T$  is even the mI-fibres begin and end on the same order 2 component of  $\Sigma_{Q'}$ , whereas when  $\bar{\alpha}^T$  is odd they run from one such component to the other. Thus in the first case the projection  $p':Q' \rightarrow B'$  folds each order 2 component onto half of  $\Sigma_{B'}$ , whereas in the second case  $p'$  maps each such component homeomorphically onto all of  $\Sigma_{B'}$ .

The special case when  $B'$  is zero is worth noting:



It looks just the same as a Type 2 fibration, except that one of the mI-fibres now consists of points which are singular of order  $\alpha'$ .

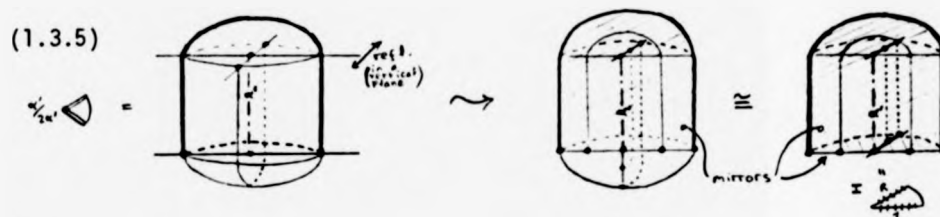
Admissible fibred involutions of  $Q' \xrightarrow{p'} B'$ , projecting to the identity or a reflection on the base, and the identity, a reflection or a rotation on the generic  $S^1$ -fibres (restricting to just the identity or a reflection on the central mI-fibre), are summarized in the following table:

(1.3.4)

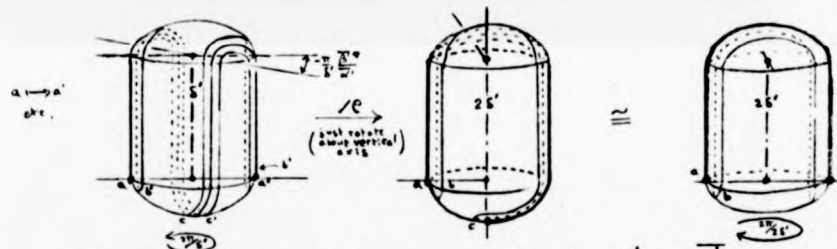
$Q'$	Action of $\rho$ on Base ( $\rho_{B'}$ )	Action of $\rho$ on Central Fibre ( $\rho_{f'}$ )	Action of $\rho$ on Generic Fibres ( $\rho_{f'}$ )	$B'$	$Q'$	$f'$	Type of $Q' \rightarrow B'$
	id.	id.	refl. (plane)				17
	id.	id.	refl. (plane)				19
	id.	refl.	refl. (plane)				18
	id.	$\begin{cases} \text{id. (if even)} \\ \text{refl. (if odd)} \end{cases}$	rot.				3
	refl.	id.	id.				10
	refl.	refl.	rot.				11
	refl.	$\begin{cases} \text{refl. (if even)} \\ \text{id. (if odd)} \end{cases}$	refl.				3



It is a routine exercise to check these, so we will only do some of the more attractive examples.



(1.3.6) Now look at  $(\rho_B, \rho_F) = (\text{id}, \text{rotn.})$ . When  $\bar{\alpha}^T$  is even, the action on the central fibre is the identity and we have:



We get  $\alpha = \alpha'$ ,  $\delta = 2\delta'$ ; so  $\bar{\alpha} = \frac{\alpha}{\delta} = \frac{\alpha'}{2\delta'} = \frac{\alpha'}{2} \bar{\alpha}'$ .

Therefore,

$$-\frac{\pi}{\delta'} \cdot \frac{\bar{\beta}^T}{\alpha'} = -\frac{\pi}{\delta} \cdot \frac{\bar{\beta}^*}{\alpha} + k_1 \frac{2\pi}{\delta} \quad (k_1 \in \mathbb{Z})$$

$$= -\frac{\pi}{2\delta'} \cdot \frac{\bar{\beta}^*}{\alpha'/2} + k_1 \cdot \frac{2\pi}{2\delta'},$$

i.e.  $\bar{\beta}^* = \bar{\beta}^T + k_1 \bar{\alpha}^T$ .

We require  $\bar{\beta}^* \bar{\beta} = 1 \pmod{\bar{\alpha}}$  i.e.  $(\bar{\beta}^T + k_1 \bar{\alpha}^T) \bar{\beta} = k_2 \cdot \frac{\bar{\alpha}^T}{2} + 1 \quad (k_2 \in \mathbb{Z})$

and also know that  $\bar{\beta}^T \bar{\beta}^* = k_3 \bar{\alpha}^T + 1 \quad (k_3 \in \mathbb{Z})$ ,

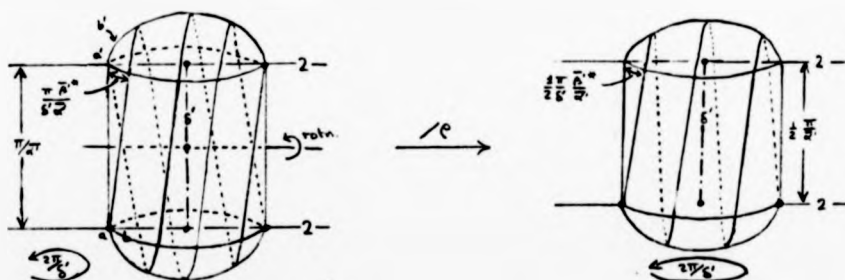
so  $\bar{\beta}^T (\bar{\beta} - \bar{\beta}^T) = \text{constant} \cdot \frac{\bar{\alpha}^T}{2}$ .

Using  $(\overline{\alpha}^T, \overline{\beta}^{T*}) = 1$  gives:  $\overline{\beta} - \overline{\beta}^T = m \frac{\overline{\alpha}^T}{2}$  ( $m \in \mathbb{Z}$ )

and so 
$$\begin{aligned} \beta &= \overline{\beta} \cdot \delta = (\overline{\beta}^T + m \frac{\overline{\alpha}^T}{2}) \cdot 2\delta' \\ &= 2\beta' + m\alpha' . \end{aligned}$$

When  $\overline{\alpha}^T$  is odd the fibred involution corresponds to rotating  $Q'$  about a suitable *horizontal* axis, giving a different calculation (with  $\alpha = \alpha'$ ,  $\delta = \delta'$ ) but the same result.

(1.3.7) Finally we look at  $(\rho_B, \rho_F) = (\text{refl.}, \text{refl.})$ . Again dividing into two cases, when  $\overline{\alpha}^T$  is even we have:



The involution corresponds to rotation about a horizontal axis; it identifies the existing order 2 singular sets and creates a new one. With the usual notation we have:

$$\delta = \delta' , \quad \alpha = 2\alpha'$$

$$\overline{\beta} = \overline{\beta}^T + m\overline{\alpha}^T \quad (\text{so } \beta = \beta' + m\alpha' , \text{ where } m \in \mathbb{Z}) .$$

Thus, in order to satisfy  $-\frac{\alpha}{2} < \beta \leq \frac{\alpha}{2}$ , the solutions are:

$$\frac{\beta}{\alpha} = \frac{\beta'}{2\alpha'} , \text{ and exactly one of } \frac{\beta' + \alpha'}{2\alpha'} , \frac{\beta' - \alpha'}{2\alpha'} .$$

Note that the action on the central fibre is reflection.



When  $\bar{\alpha}^T$  is odd we rotate about a vertical axis passing along the central fibre and obtain:

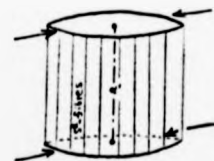
$$\delta = 2\delta', \quad \alpha = 2\alpha' \quad (\text{so } \bar{\alpha} = \bar{\alpha}'),$$

$$\bar{\beta} = \frac{\bar{\beta}^T + m\bar{\alpha}^T}{2} \quad (\text{so } \beta = \beta' + m\alpha', \text{ where } m \in \mathbb{Z}),$$

yielding the same solution. The action on the central fibre is the identity.

(1.3.8) It remains to show that Table (1.3.4) is complete. This is a similar exercise to that which we performed for the Type 1 fibrations; essentially, it involves showing that once the fibres are "twisted", certain maps do not make sense - the interested reader can check this at her (or his) leisure.

(1.4) Type 4:   $Q' \cong (D^2 \times S^1) / \mathbb{Z}_{2\alpha'}$   
 $\downarrow$   $\downarrow$   $\downarrow$   
  $B' \cong D^2 / \mathbb{Z}_{2\alpha'}$



The generator of  $\mathbb{Z}_{2\alpha'}$  acts by rotation through  $\pi/\alpha'$  on  $D^2$  and reflection on  $S^1$ ; thus  $Q'$  takes the form of a cylinder with points on the top disc identified by the antipodal map about its centre (or, equivalently, by a rotation), and similarly for the bottom disc. The central fibre is therefore an  $mI$ -fibre and is singular of order  $\alpha'$ . Its two endpoints are non-manifold points (they have neighbourhoods with

underlying space homeomorphic to the cone on  $\mathbb{RP}^2$  ; it is surrounded by (generic)  $S^1$ -fibres and is the first non-orientable local fibre type.

The table of objects of which it forms the fibred double cover is as follows:

(1.4.1)

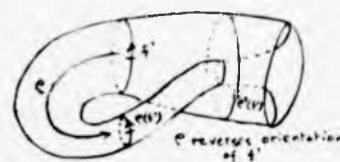
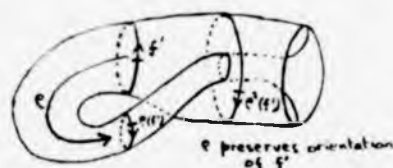
Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Central Fibre	Action of $\rho$ on $S^1$ -fibres( $\rho_{f_i}$ )	B	Q	f	Type of $Q \rightarrow B$
id.	id.	refl.			mI	13
id.	refl.	refl.				14
id.	refl.	rotn.				4
refl.	id.	-			S <sup>1</sup>	10
refl.	refl.	-				11

That these are as stated is an easy exercise. The omission of the action on the generic fibres for the last two is because it can be viewed as a rotation or a reflection, owing to the non-orientable nature of  $Q'$  ; the action on the base and the central fibre is enough to fully specify  $\rho$  .

To ensure that this list is complete, we must also prove:

(1.4.2) Lemma: There is no involution  $\rho$  whose projection to the base is a rotation.

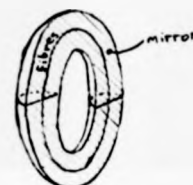
Proof: Consider the action of  $\rho$  on  $aQ^T$  , which is a Klein bottle fibred by orientation-preserving circles. We have one of the following two pictures:



Whichever way  $\rho$  maps  $f'$ , although  $f'$  and  $\rho^2(f')$  may coincide (we have shown them as separate here for the sake of clarity), we cannot have  $\rho^2 = \text{identity}$  as required since the sense of  $f'$  is reversed in both cases.  $\square$

(1.5) Type 5:

$$\begin{array}{ccc} \text{Diagram of } Q' & Q' \cong (D^2 \times S^1)/\mathbb{Z}_2 & \\ \downarrow p' & & \downarrow \\ \text{Diagram of } B' & B' \cong D^2/\mathbb{Z}_2 & \end{array}$$



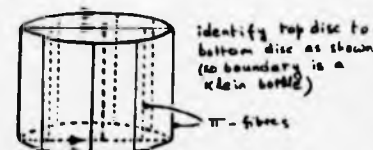
$\mathbb{Z}_2$  acts by reflection on the base and the identity on the fibres.

All the obvious combinations of action are admissible and we get the following table:











Action of $\rho$ on Base ( $\rho_B$ )	Action of $\rho$ on Regular Fibres ( $\rho_F$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	rotn.			$S^1$	5
id.	refl.			$mI$	15
refl.	id.			$S^1$	7
refl.	rotn.				9
refl.	refl.				10



(1.6) Type 6:

$$\begin{array}{ccc} \text{Diagram of } Q' & Q' \cong (D^2 \times S^1)/\mathbb{Z}_2 & \\ \downarrow p' & & \downarrow \\ \text{Diagram of } B' & B' \cong D^2/\mathbb{Z}_2 & \end{array}$$















$\mathbb{Z}_2$  acts by reflection on the base and rotation on the fibres; consequently all the fibres are circles, even those over  $\partial|B'|$ . All the obvious involutions are admissible:

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Regular Fibres( $\rho_F$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	rotn.			$S^1$	5
id.	refl.			$mI$	16
refl.	id.			$S^1$	9
refl.	rotn.				8
refl.	refl.				11

(1.7) Type 7:   $Q' \cong (D^2 \times S^1) / \Delta_{2\alpha'}$   
 $\downarrow$   $p' \downarrow$   $\downarrow$   
  $B' \cong D^2 / \Delta_{2\alpha'}$

The generator of  $\mathbb{Z}_{\alpha'} \leq \Delta_{2\alpha'}$  acts by rotation on the base and the identity map on fibres, whilst each element of  $\Delta_{2\alpha'} - \mathbb{Z}_{\alpha'}$  reflects the base and is the identity on fibres. The picture is therefore exactly as for Type 5 (1.5) except that the central fibre is now singular of order  $\alpha'$ . Hence the objects covered by Type 7 fibrations can easily be derived from those covered by Type 5 fibrations. We assume  $\alpha' \geq 2$ .

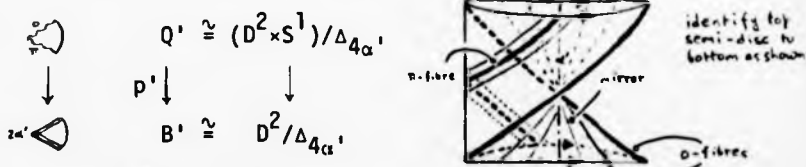
Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Regular Fibres( $\rho_F$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	rotn.			$S^1$	7
id.	refl.			$mI$	17
refl.	id.			$S^1$	7
refl.	rotn.				9
refl.	refl.				10

(1.8) Type 8:   $Q' \cong (D^2 \times S^1) / \Delta_{2\alpha'}$   
 $\downarrow$   $p' \downarrow$   $\downarrow$   
  $B' \cong D^2 / \Delta_{2\alpha'}$

The generator of  $\mathbb{Z}_{\alpha'} \leq \Delta_{2\alpha'}$  acts by rotation on the base of  $D^2 \times S^1$  and by the identity on fibres, whilst elements of  $\Delta_{2\alpha'} - \mathbb{Z}_{\alpha'}$  act by reflection on the base and rotation of fibres. Analogously with the relationship of Type 7 to type 5, Type 8 is just a more general version of Type 6 with the central fibre now being singular of order  $\alpha'$ . Again,  $\alpha' \geq 2$  here.

Action of $\rho$ on $\text{Base}(\rho_B)$	Action of $\rho$ on Regular Fibres( $\rho_F$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	rotn.			$S^1$	7
id.	refl.			mI	18
refl.	id.			$S^1$	9
refl.	rotn.				8
refl.	refl.				11

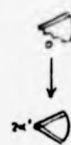
(1.9) Type 9:



The generator of  $\mathbb{Z}_{2\alpha'}$  rotates the base through  $\pi/\alpha'$  and the fibres through  $\pi$ . Elements of  $\Delta_{4\alpha'} - \mathbb{Z}_{2\alpha'}$  act by reflection on the base and identity/reflection on fibres; we saw the nature of this in (1.1.6/7). An  $S^1$ -fibre (singular of order  $\alpha'$ ) rises up the centre of the figure, the Type 5 fibres ("O-fibres") make up  $\partial|Q'|$ , and the Type 6 fibres (" $\pi$ -fibres") lie in the interior. A fibred involution cannot map distinct fibre types to one another and so must project to the base as the identity map.

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Fibres( $\rho_f$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	rotn.			$S^1$	7
id.	refl.			mI	19

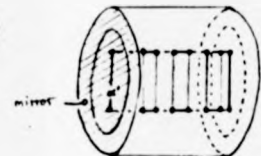
(1.10) Type 10:



$$Q' \cong (D^2 \times S^1) / (\Delta_{2\alpha'} \times \mathbb{Z}_2)$$

$$p' \downarrow$$

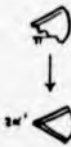
$$B' \cong D^2 / \Delta_{4\alpha'}$$



This is just Type 3 with slope  $0/\alpha'$  (see 1.3.3) factored out by a reflection in a vertical plane (producing a mirror). Again, the only operation on the base is the identity map.

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on mI-fibres	Action of $\rho$ on Generic Fibres( $\rho_f$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	id.	refl.			mI	17
id.	refl.	refl.				19
id.	refl.	rotn.			$S^1$	10

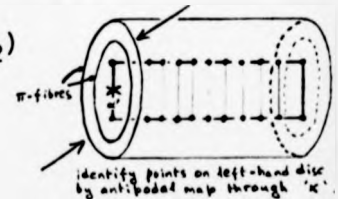
(1.11) Type 11:



$$Q' \cong (D^2 \times S^1) / (\Delta_{2\alpha'} \times \mathbb{Z}_2)$$

$$p' \downarrow$$

$$B' \cong D^2 / \Delta_{4\alpha'}$$



Just as for the previous type, this is produced by factoring out a Type 3 fibration with zero slope. This time however,  $\mathbb{Z}_2$  acts by reflecting the base and simultaneously rotating the  $S^1$ -fibres through half their length. Thus we have no mirror, but instead get a non-manifold point (marked 'x') in the underlying space.



Action on $\rho$ on Base( $\rho_{B'}$ )	Actions of $\rho$ on mI-fibres	Action of $\rho$ on Generic Fibres ( $\rho_{f'}$ )	$B$	$Q$	$f$	Type of $Q \rightarrow B$
id.	id.	refl.			} mI $S^1$	19
id.	refl.	refl.				18
id.	refl.	rotn.				10

From here onwards (Types 12-19), the generic fibre of  $Q' \xrightarrow{p'} B'$  is always mI and hence so is the generic fibre of  $Q \xrightarrow{p} B$ .

(1.12) Type 12:  $Q' \cong (D^2 \times S^1)/\mathbb{Z}_2$   
 $p' \downarrow$   $B' \cong D^2$

$\mathbb{Z}_2$  acts simply by reflection on the fibres and so  $Q'$  is just a cylinder with mirrors at both ends. The following list of coverings is easily derived:

Action of $\rho$ on Base( $\rho_{B'}$ )	Action of $\rho$ on Fibres( $\rho_{f'}$ )	$B$	$Q$	Type of $Q \rightarrow B$
id.	refl.			12
rotn.	id.			13
rotn.	refl.			14
refl.	id.			15
refl.	refl.			16

(1.13) Type 13:  $Q' \cong (D^2 \times S^1)/(\mathbb{Z}_{\alpha'} \times \mathbb{Z}_2)$   
 $p' \downarrow$   $B' \cong D^2/\mathbb{Z}_{\alpha'}$

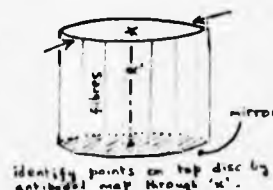


This is just Type 12 with the central fibre replaced by one which is singular of order  $\alpha'$ .

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Fibres( $\rho_F$ )	B	Q	Type of Q $\rightarrow$ B
id.	refl.			13
rotn.	id.			13
rotn.	refl.			14
refl.	id.			17
refl.	refl.			18

(1.14) Type 14:

$$\begin{array}{ccc}
 \text{cone with dot} & Q' \cong (D^2 \times S^1) / (\mathbb{Z}_{2\alpha'} \times \mathbb{Z}_2) & \\
 \downarrow p' & & \\
 \text{cone with dot} & B' \cong D^2 / \mathbb{Z}_{2\alpha'} &
 \end{array}$$







The generator of  $\mathbb{Z}_{2\alpha'}$  acts by rotation on the base (through  $\pi/\alpha'$ ) and on the fibres (through  $\pi$ ).  $\mathbb{Z}_2$  simply reflects the fibres. Thus we have a Type 4 fibration (1.4) modulo a reflection in a plane perpendicular to its fibres, yielding the mirror on the base of our figure. Consequently there is now only one non-manifold point of the underlying space, marked 'x'.

The boundary of  $Q'$  is an mI-fibred Mobius band and so, by a similar argument to that of Lemma (1.4.2), the restriction to the base of any fibred involution  $\rho: Q' \rightarrow Q'$  can be only the identity or a reflection but not a rotation. Moreover there is only one such map for which  $\rho_B$  is a reflection. For suppose  $f_1$  and  $f_2$  are the fibres which are mapped to themselves by  $\rho$ :

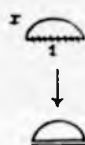


Further suppose that an endpoint  $a$  of  $f_1$  is fixed by  $\rho$ . Then, since

$\rho$  is continuous and non-trivial, it must exchange the endpoints  $c$  and  $c'$  of  $f_2$  and in fact be a reflection in a vertical plane. If  $a$  is not fixed then  $c$  and  $c'$  must be, and we have the same case.

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Central Fibre	Action of $\rho$ on Generic Fibres( $\rho_F$ )	$\underline{B}$	$\underline{Q}$	Type of $Q \rightarrow B$
id.	id.	refl.			13
refl.	id.	-			19

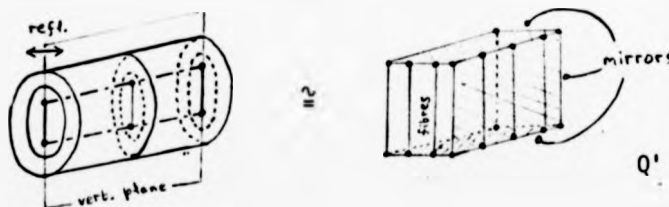
(1.15) Type 15:









$$Q' \cong (D^2 \times S^1) / (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

$$\begin{array}{ccc} p' \downarrow & & \downarrow \\ B' \cong \mathbb{Z}^2 & & D^2 / \mathbb{Z}_2 \end{array}$$

$Q'$  is formed by a reflection of a Type 2 fibration in the vertical plane in which its band of mI-fibres lies:

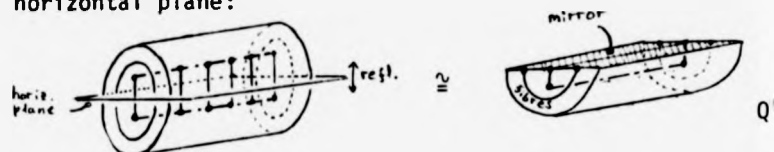


Thus  $Q'$  consists of a vertically fibred parallelopiped which has mirrors on the top and bottom as well as on one face.

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Fibres( $\rho_F$ )	$\underline{B}$	$\underline{Q}$	Type of $Q \rightarrow B$
id.	refl.			15
refl.	id.			17
refl.	refl.			19

(1.16) Type 16:  $\begin{array}{c} \text{---} \\ \downarrow \end{array}$   $\begin{array}{c} Q \cong (D^2 \times S^1) / (\mathbb{Z}_2 \times \mathbb{Z}_2) \\ \downarrow p' \\ B' \cong D^2 / \mathbb{Z}_2 \end{array}$

$Q'$  is again formed by reflecting a Type 2 fibration, but this time in a horizontal plane:



A band of  $mI$  fibres is surrounded by half-cylinders of  $mI$ -fibres folding onto it. The flat surface on top of the picture is a mirror.

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Fibres( $\rho_F$ )	$B$	$Q$	Type of $Q \rightarrow B$
id.	refl.			15
refl.	id.			19
refl.	refl.			18

(1.17) Type 17:  $\begin{array}{c} \text{---} \\ \downarrow \end{array}$   $\begin{array}{c} Q' \cong (D^2 \times S^1) / (\Delta_{2\alpha} \times \mathbb{Z}_2) \\ \downarrow p' \\ B' \cong D^2 / \Delta_{2\alpha} \end{array}$



This is just Type 15 (1.15) with one fibre (in  $\partial|Q'|$ ) made singular of order  $\alpha'$

Action of $\rho$ on Base( $\rho_B$ )	Action of $\rho$ on Fibres( $\rho_F$ )	$B$	$Q$	Type of $Q \rightarrow B$
id.	refl.			17
refl.	id.			17
refl.	refl.			19

(1.18) Type 18:



$$Q' \cong (D^2 \times S^1) / (\Delta_{2\alpha'} \times \mathbb{Z}_2)$$

$$p' \downarrow \quad \downarrow$$

$$B' \cong D^2 / \Delta_{2\alpha'}$$



This is just Type 16 (1.16) with one of the  $mI$ -fibres in the vertical band made singular of order  $\alpha'$ .

Action of  $\rho$   
on Base( $\rho_B$ )

Action of  $\rho$   
on Fibres( $\rho_F$ )

id.

refl.

refl.

id.

refl.

refl.

B



Q



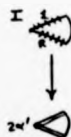
Type of  
 $Q \rightarrow B$

17

19

18

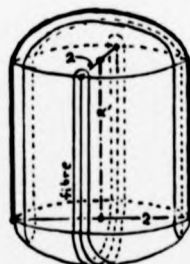
(1.19) Type 19:



$$Q' \cong (D^2 \times S^1) / (\Delta_{4\alpha'} \times \mathbb{Z}_2)$$

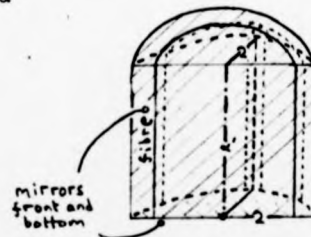
$$p' \downarrow \quad \downarrow$$

$$B' \cong D^2 / \Delta_{4\alpha'}$$



refl.

III



mirrors  
front and  
bottom

$Q'$  is isomorphic to a Type 3 fibration of slope  $\alpha'/2\alpha'$  modulo a reflection in a vertical plane. The only fibred involution is one which projects to the identity map on the base.

Action of  $\rho$  on Base( $\rho_B$ )

Action of  $\rho$  on Fibres( $\rho_F$ )

B

Q

Type of  
 $Q \rightarrow B$

id.

refl.



17

Table 3. A Complete List of Local Fibred Double Covers

Local fibre types  $Q \rightarrow E$  on the left are double covered by local fibre types  $Q' \rightarrow E'$  on the right. The table is split into five sections (Tables 3.1 to 3.5) depending on the nature of  $\bar{f}: E' \rightarrow E$  (the projection of the covering map to the base spaces) and the generic fibres.

First we have Tables 3.1 and 3.2, for which  $\bar{f}: E' \rightarrow E$  is a double covering:

Table 3.1 $\bar{f}: E' \rightarrow E$ is a double covering and the generic fibres are $S^1$ 's.							
Type of $Q \rightarrow B$	$f$	$Q$	$B$	$f'$	$Q'$	$B'$	Type of $Q' \rightarrow E'$
1	$S^1$			$S^1$	and		0, 1
2	$S^1$			$S^1$			0
3	$S^1$			$S^1$			1
					and		2, 3
4	$S^1$			$S^1$			1
5	$S^1$			$S^1$			0
6	$S^1$			$S^1$			0
7	$S^1$			$S^1$			1
							5, 7
8	$S^1$			$S^1$			1
							6, 8
9	$S^1$			$S^1$			1
							5, 7
							6, 8
10	$S^1$			$S^1$			2, 3
							4
							5, 7
11	$S^1$			$S^1$			2, 3
							4
							6, 8

Table 3.2 :  $\tilde{f}: B' \rightarrow B$  is a double covering and the generic fibres are  $mI$ 's.

Type of $Q \rightarrow B$	$f$	$Q$	$B$	$f'$	$Q'$	$B'$	Type of $Q' \rightarrow B'$
13	$mI$			$mI$			12, 13
14	$mI$			$mI$			12, 13
15	$mI$			$mI$			12
16	$mI$			$mI$			12
17	$mI$			$mI$			13
18	$mI$			$mI$			15, 17
19	$mI$			$mI$			13
19	$mI$			$mI$			16, 18

Tables 3.3 to 3.5 deal with those coverings for which  $\tilde{p}: B' \rightarrow B$  is a homeomorphism:

Table 3.3 :  $\tilde{f}: B' \xrightarrow{\sim} B$ ,  $Q \rightarrow B$  and  $Q' \rightarrow B'$  both have generic fibre  $S^1$ .

Type of $Q \rightarrow B$	$f$	$Q$	$B$	$f'$	$Q'$	$B' (\cong B)$	Type of $Q' \rightarrow B'$
0	$S^1$			$S^1$			0
1	$S^1$			$S^1$			1
2	$S^1$			$S^1$			2
3	$S^1$			$S^1$			3
4	$S^1$			$S^1$			4
5	$S^1$			$S^1$			5
6	$S^1$			$S^1$			6
7	$S^1$			$S^1$			7
8	$S^1$			$S^1$			8
9	$S^1$			$S^1$			9
10	$S^1$			$S^1$			10
11	$S^1$			$S^1$			11

Table 3.4 : $f: B' \xrightarrow{\sim} B$ , $Q \rightarrow B$ has generic fibre $mI$ and $Q' \rightarrow B'$ has generic fibre $S^1$ .							
Type of $Q \rightarrow B$	$f$	$Q$	$B$	$f'$	$Q'$	$B' (\cong B)$	Type of $Q' \rightarrow B'$
12	$mI$			$S^1$			0
13	$mI$			$S^1$			1
							4
14	$mI$			$S^1$			1
							4
15	$mI$			$S^1$			2
16	$mI$			$S^1$			2
							6
17	$mI$			$S^1$			3
							7
							10
18	$mI$			$S^1$			3
							8
							11
19	$mI$			$S^1$			3
							9
							10
							11

Table 3.5 : $f: B' \xrightarrow{\sim} B$ , $Q \rightarrow B$ and $Q' \rightarrow B'$ both have generic fibre $mI$ .							
Type of $Q \rightarrow B$	$f$	$Q$	$B$	$f'$	$Q'$	$B' (\cong B)$	Type of $Q' \rightarrow B'$
12	$mI$			$mI$			12
13	$mI$			$mI$			13
							14
15	$mI$			$mI$			15
17	$mI$			$mI$			17
							18
							19



## §2. Classification of $S^1$ -Fibred 3-Crystals.

Having first determined the local fibre types in  $[B + S]$ , Bonahon and Siebenmann then use them to classify the compact connected  $S^1$ -fibred 3-orbifolds  $Q \rightarrow B$ , dividing the problem into three cases:

- (a)  $Q$  is orientable (and assumed to be oriented) - note that the generic fibre must be  $S^1$ .
- (b)  $Q$  is non-orientable with generic fibre  $S^1$ .
- (c)  $Q$  is non-orientable with generic fibre  $m\mathbb{I}$ .

The base-free classification of such objects involves the assignment of a "weak data function"  $\psi$  to  $B$ , completely specifying the fibration up to equivalence of such functions. This equivalence takes the form of permissible changes ("moves") which may be made to  $\psi$  whilst leaving  $Q \rightarrow B$  unaltered up to fibred isomorphism (orientation-preserving, where appropriate).

The following is a synopsis of the classification in  $[B + S]$ , amalgamating (a) and (b) into a single case for generic fibre  $S^1$ , together with an explanation of why the data function takes the form that it does. First we need a new term:

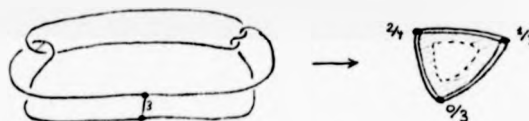
(2.0) Definition: Given an  $S^1$ -fibred 3-orbifold  $Q \xrightarrow{p} B$  and a mirror cycle  $C \subseteq B$  (0.4.2), if  $N(C)$  is a regular neighbourhood of  $C$  in  $B$

such that  $p^{-1}N(C)$  is built entirely of Type 2 and Type 3 fibrations then we refer to the singular subset of  $p^{-1}N(C)$ , i.e.

$$\xi = \Sigma_Q \cap p^{-1}N(C) ,$$

as a *Montesinos cycle*. (Strictly this is a *generalized Montesinos cycle*  $[B + S]$ , but we make no distinction here.) Sometimes  $C$  will be referred to as a Montesinos cycle, meaning that  $p^{-1}(C)$  is. //

(2.0.1) Example:



//

(2.1) Classification when the Generic Fibre is  $S^1$ .

(2.1.1)  $\psi$  assigns to  $B$  the Type Diagram for  $Q$ .

The *type diagram* takes the form of a picture of  $B$  marked to indicate the local fibre types of  $Q$  over  $\Sigma_B$ ; since the generic fibre is  $S^1$  these must be of Types 0 to 11. We will have occasion to refer to the *weak type diagram* of  $Q \rightarrow B$ , which is the same thing except that the distinction between Type 5 and Type 6 fibrations is ignored i.e.

 and  are drawn just as .

(2.1.2)  $\psi$  marks by  $**$ 's a finite number  $n_s$  of points in  $\partial|B|$ ; these are not dihedral points and there is at most one per component of  $\partial|B|$ .  $n_s$  is congruent mod 2 to the number  $n_4$  of Type 4 cone points.

$Q$  can be non-orientable by virtue of having non-orientable local fibrations or because it is glued together, over non-trivial loops in  $B$ , in a non-orientable way (or both). The latter can be depicted by indicating a minimal collection of disjoint 1-submanifolds  $K$  in  $|B| - \text{int } N(X_4)$  along which one must cut to make  $p^{-1}(\text{int}|B| - N(X_4) - K)$  orientable, where  $X_4$  is the set of Type 4 cone points. Since the class of  $K$  in

$$H_1(|B| - \text{int } N(X_4), \text{ boundary}; \mathbb{Z}_2)$$

is well-defined and since  $\partial N(X_4)$  represents a collection of orientation-reversing paths, we can ensure that  $K$  meets any given boundary component at most once, and each component of  $\partial N(X_4)$  exactly once. Thus, in the base-free classification, it suffices to mark the appropriate components of  $\partial|B|$  with a '\*'.

(2.1.3) When  $n_s = n_4 = 0$ ,  $\Psi$  assigns to  $B$  one of the Seifert Types  $Q$ ,  $N_0$ ,  $N_{nI}$ ,  $N_{nII}$ ,  $N_{nIII}$  that can be realized by  $B$ .

We need only assign a Seifert type when  $\partial|K| \neq \emptyset$  in (2.1.2); then either  $K = \emptyset$  or it consists of a collection of closed paths. By fixing a base point in  $B$ , these can be amalgamated to form a single such path which we shall also call  $K$ .  $\Psi$  then assigns Seifert types [Sei] as follows:

$$\begin{aligned} \underline{Q} &: K = \emptyset \\ N_0 &: K \neq \emptyset, |B| \text{ is orientable.} \end{aligned}$$

$N_n I$  :  $K \neq \emptyset$  ,  $|B|$  is non-orientable,  $|B|-K$  is orientable.

$N_n II$  :  $K \neq \emptyset$  ,  $|B|$  is non-orientable,  $|B|-K$  is non-orientable,  
 $K$  is 1-sided.

$N_n III$  :  $K \neq \emptyset$  ,  $|B|$  is non-orientable,  $|B|-K$  is non-orientable,  
 $K$  is 2-sided.

Note that having Seifert type  $\underline{0}$  (i.e.  $K = \emptyset$ ) does not by itself imply that  $Q$  is orientable - there must also be no non-orientable fibre types specified in (2.1.1).

Obviously, whether a given  $B$  has  $|B|$  orientable or not restricts the possible Seifert types. There are further restrictions. If  $\bar{B}$  denotes  $|B|$  with its boundary components closed off by discs then, when  $\bar{B}$  is a 2-sphere,  $\psi$  can only assign Seifert type  $\underline{0}$  , since any suitable non-empty  $K$  would have to run between components of  $\partial|B-\text{int } N(X_4)|$  and would be dealt with by (2.1.2). If  $\bar{B} \cong \mathbb{RP}^2$  then only  $\underline{0}$  and  $N_n I$  can occur, since cutting  $\mathbb{RP}^2$  along a closed path always yields an orientable component. If  $\bar{B}$  is a Klein bottle we cannot have  $N_n III$  , since cutting along a 2-sided loop leaves an orientable component.

(2.1.4)  $\psi$  assigns slopes  $\beta/\alpha \in \mathbb{Q}$  to the Type 1 cone points of angle  
 $2\pi/\alpha$  and the Type 3 dihedral points of angle  $\pi/\alpha$  in  $B$  .

These fibrations were explained in (1.1) and (1.3);  $\beta/\alpha$  records how "twisted" they are. We are *not* restricting  $\beta/\alpha$  to lie in  $(-\frac{1}{2}, \frac{1}{2}]$  here.

(2.1.5)  $\psi$  assigns a parity  $\epsilon \in \mathbb{Z}_2$  to each Montesinos cycle  $C$  in  $\partial|B|$ .

The bands of  $mI$ -fibres coming from the Type 2 fibration pieces of  $p^{-1}(C)$  can if necessary be extended over the dihedral points to form either an annulus or a Möbius band, denoted by  $\epsilon = 0$  or  $1$  respectively. This parity can be altered by extending differently over any given corner, corresponding to a unit change in slope there. However, if there are no corners in  $B$  then  $\epsilon(C)$  is fixed and indicates that  $p^{-1}(C)$  is an actual annulus or Möbius band, embedded as a 2-suborbifold in  $Q$ .

(2.1.6) When  $B$  is closed and  $Q$  is orientable (i.e. no non-orientable fibre types assigned in (2.1.1), no '\*' 's on  $\partial|B|$ , and  $B$  has Seifert type  $Q$  in (2.1.3)) then  $\psi$  assigns an Euler number  $e_0 \in \mathbb{Q}$  to  $B$ , subject to:

$$e_0 + \sum \{m(x) \frac{\beta(x)}{\alpha(x)} : x \in X^0\} + \sum \{\frac{1}{2}\epsilon(C) : C \text{ is a mirror cycle}\} \in \mathbb{Z},$$

$$\text{where } m(x) = \begin{cases} 1 & \text{when } x \text{ is conical. } (\tau_{31} = 1) \\ \frac{1}{2} & \text{when } x \text{ is dihedral. } (\tau_{31} = 3) \end{cases}$$

$X^0$  is the set of cone and dihedral points in  $B$ .  $e_0$  represents the obstruction to finding a global (n-fold) section of  $Q \rightarrow B$  and corresponds to the classical Euler number for fibre bundles. For  $Q$  to be euclidean,  $e_0$  must be zero.

(2.1.7) When  $n_4 = n_5 = 0$ , the Seifert type is not  $Q$ , and  $\partial|B|$  consists only of Montesinos cycles  $C_i$  or is empty, then  $\psi$  assigns to each  $C_i$  an integer  $\epsilon'(C_i) \in \mathbb{Z}$  of the same parity as  $\epsilon(C_i)$ , and a parity  $n \in \mathbb{Z}_2$  to  $B$ .

Since the Seifert type is not 0 here, this applies only when  $Q$  is non-orientable (though all the local fibre types of  $Q$  are orientable).

If  $\partial|B| \neq \emptyset$ , choose a 1-fold section  $s$  of  $p$  over  $\partial N(\partial|B|)$ , together with the standard 2-fold section  $t$  of  $p$  over

$$N(\partial|B|) = \{\text{corners}\}$$

having slope zero over  $\partial N(x)$ , for each corner  $x$ . Then if  $s, t$  restrict to  $s_i, t_i$  over  $\partial N(C_i)$  they have intersection number  $\epsilon'(C_i) = s_i \cdot t_i \in \mathbb{Z}$  in  $p^{-1}(\partial N(C_i))$ , with the same parity as  $\epsilon(C_i)$ ;  $n \in \mathbb{Z}_2$  is the obstruction to extending  $s$  to a 1-fold section of  $p$  over  $B-N(X^0)$  so that it has slope zero over  $\partial N(x)$  for each cone point  $x \in B$ . Changing the choice of  $s$  will of course change  $\epsilon'$  and  $n$ .

If  $\partial|B| = \emptyset$  then we have only  $n$  to define and we do so in the obvious way:  $n = 0$  if there is a 1-fold section over  $B-N(X^0)$  having slope zero over  $\partial N(x)$  for each cone point  $x \in B$ ; otherwise  $n = 1$ . We can alter  $n$  by changing the parameterization (hence the slope) at a cone point.

N.B. This case - when  $\partial|B| = \emptyset$  - seems to have been ignored in  $[B + S]$ .

In (2.1.1) to (2.1.7) we have shown how to represent diagrammatically an  $S^1$ -fibred 3-orbifold  $Q \rightarrow B$  (with generic fibre  $S^1$ ) using a "weak data function". We now need to specify when two such functions represent the same (i.e. a fibred-isomorphic) orbifold.

(2.1.8) Weak data functions  $\psi$  are equivalent up to the following moves, which neutralize the choices of parameterizations, orientations, etc:

- (a) Simultaneously change the signs of all slopes  $\beta/\alpha \in \mathbb{Q}$ , all integers  $\epsilon' \in \mathbb{Z}$ , and  $e_0 \in \mathbb{Q}$ , wherever these are defined.
  - (b<sub>1</sub>) Change the slope  $\beta/\alpha$  at a conical point to  $\beta/\alpha + n$  with  $n \in \mathbb{Z}$ . When  $n \in \mathbb{Z}_2$  is defined, simultaneously change it to  $n + n$ .
  - (b<sub>2</sub>) Change the slope  $\beta/\alpha$  at a dihedral point to  $\beta/\alpha + n$ , with  $n \in \mathbb{Z}$ . When it is contained in a Montesinos cycle, simultaneously change the parity  $\epsilon \in \mathbb{Z}_2$  to  $\epsilon + n$  and, when defined,  $\epsilon' \in \mathbb{Z}$  to  $\epsilon' + n$ .
  - (c) For a Montesinos cycle  $C$ , replace the weight  $\epsilon' \in \mathbb{Z}$  (when defined) by  $\epsilon' + 2$  and simultaneously change  $n \in \mathbb{Z}_2$  to  $n + 1$ .
- When  $B$  is not assigned Seifert type 0 in (2.1.3) we also have the following three moves:
- (d<sub>1</sub>) Slide a '\*' in  $\partial|B|$  over a dihedral point  $x$ , simultaneously reversing the sign of the slope at  $x$  from  $\beta/\alpha \in \mathbb{Q}$  to  $-\beta/\alpha \in \mathbb{Q}$ .
  - (d<sub>2</sub>) For any conical point with slope  $\beta/\alpha$ , reverse it to  $-\beta/\alpha$ .
  - (d<sub>3</sub>) On any component  $C$  of  $\partial|B|$ , simultaneously reverse the signs of all slopes  $\beta/\alpha \in \mathbb{Q}$  (when defined) and of the weight  $\epsilon' \in \mathbb{Z}$  (when defined).

---

Note that, using (b<sub>1</sub>) and (b<sub>2</sub>), every data function can be normalized within its equivalence class so that all slopes  $\beta/\alpha \in \mathbb{Q}$  lie in the interval  $(-\frac{1}{2}, \frac{1}{2}]$ ; if  $\psi$  does not assign Seifert type 0 to  $B$  in (2.1.3) (i.e. it assigns one of the other four Seifert types, or

$n_5 + n_4 > 0$ ), all slopes  $\beta/\alpha$  can be chosen to lie in the interval  $[0, \frac{1}{2}]$ . Using (c), every  $\epsilon'(C_i)$  can be assumed to be equal to  $\epsilon(C_i)$ . We shall adopt these conventions as a matter of course.

(2.2) Classification when the Generic Fibre is  $mI$

(2.2.1)  $\psi$  assigns to  $B$  the type diagram for  $Q$ .

Local fibrations must be of Types 12-19.

(2.2.2)  $\psi$  marks a finite number  $n_s$  of components of  $\partial|B|$  by '\*' 's, not more than one per component.  $n_s$  must be congruent mod 2 to the number of Type 14 points assigned by (2.2.1).

Just as for when the generic fibres are circles, the '\*' 's relate to the homology class of the 1-submanifold  $K$  along which we cut to make

$$p^{-1}(\text{int } |B| - N(X_{14}) - K)$$

orientable, where  $X_{14}$  is now the set of Type 14 cone points.

(2.2.3) When  $X_{14} = \emptyset$  and no '\*' 's are assigned in (2.2.2),  $\psi$  gives  $B$  one of the Seifert types  $0$ ,  $N_0$ ,  $N_n I$ ,  $N_n II$ ,  $N_n III$ .

As in (2.1.3), this tells us that the submanifold  $K$  (cf 2.2.2) is a loop in  $\text{int } |B|$ ; the Seifert types are defined as before.

---

Equivalence of weak data functions  $\psi$  when the generic fibre is  $mI$  is simply by equality, since no choices were made when producing  $\psi$ .

---



Fortunately, much of this panoply of information is superfluous to our needs, for we wish to look at only the  $S^1$ -fibred 3-crystals i.e. our orbifolds must be euclidean. Proposition (0.7.3) tells us that we are looking for all those  $Q \xrightarrow{P} B$  for which  $B$  is euclidean and  $e_0(p) = 0$  (see 0.6.2). Thus we must classify fibrations  $Q \xrightarrow{P} B$  over the 17 closed euclidean 2-orbifolds i.e. the 2-crystals (Table 2). When  $Q$  is orientable we are subject to the further restriction that  $e_0(p) = 0$  (this is automatic, by definition, when  $Q$  is non-orientable). The classification of  $S^1$ -fibred 3-orbifolds in this restricted case boils down to:

(2.3) Fibre  $S^1$

- (i) Assign a type diagram to  $B$ , where  $B$  is one of the 17 2-crystals.
- (ii) Mark by '\*' 's a finite number  $n_s$  of points in  $\partial|B|$ , not corners, with at most one '\*' per component of  $\partial|B|$ , such that  $n_s \equiv n_4 \pmod{2}$ , where  $n_4$  is the number of type 4 points assigned in (i).
- (ii') When  $n_s = n_4 = 0$ , assign one of the Seifert types to  $B$  subject to:

$ B $	Permissible Seifert Types
$S^2$	$\underline{0}$
$D^2$	$\underline{0}$
Ann	$\underline{0}$
$T^2$	$\underline{0}, N_0$
$IRP^2$	$\underline{0}, N_n I$
Möb	$\underline{0}, N_n I$
$K^2$	$\underline{0}, N_n I, N_n II$

- (iii) Assign slopes  $\beta/\alpha \in (-\frac{1}{2}, \frac{1}{2}]$  to each Type 1 and Type 3 point in  $B$  and parities  $\epsilon_i$  to each Montesinos cycle  $C_i$  in  $B$ ; when  $Q$  is orientable (i.e. no non-orientable local fibre types in (i), no '\*' 's in (ii) and Seifert type 0 in (ii')) then these are subject to:

$$\sum \{m(x) \frac{\beta(x)}{\alpha(x)}\} + \sum \{\frac{1}{2}\epsilon(C_i)\} \in \mathbb{Z} ,$$

where  $m(x)$  is 1 for cone points and  $\frac{1}{2}$  for dihedral points - see (2.1.6). Note that if there is only one Montesinos cycle  $C_1$  then its parity is automatically determined - when drawing such objects we therefore write the parity in parentheses or even omit it entirely.

- (iv) When  $n_5 = n_4 = 0$  and (ii') assigns Seifert type other than 0 then assign a weight  $\eta \in \mathbb{Z}_2$  to  $B$  (we will assume that  $\epsilon'(C_i) = \epsilon(C_i)$  for all  $C_i$  - see (2.1.7)).

#### (2.4) Fibre mI :

- (i) Assign a type diagram to one of the 17 2-crystals  $B$  .  
 (ii) Mark certain components by '\*' 's as before so that  $n_5 \equiv n_{14} \pmod{2}$ , where  $n_{14}$  is the number of type 14 points assigned in (i).  
 (ii') When no '\*' 's occur in (ii) give  $B$  a Seifert type as per the list for fibre  $S^1$  .

(2.5) Using (2.3) and (2.4) we can now give the complete list of  $S^1$ -fibred 3-crystals as indicated in  $[B + S]$ . Since Bonahon and Siebenmann's methods are purely topological, they offer no direct way of knowing to which crystallo-

Table 4 : The Crystallographic Space Groups and their  $S^1$ -Fibred 3-Crystals.

Int'l No. Int'l Symbol

Fibred 3-Crystal(s)

Triclinic  $G_m = C_1$  ;  $G_m^1 = \{id\}$  ,  $B = \text{---}$

1 P1  Type 0 (many)

Triclinic  $G_m = \bar{C}_2$  ;  $G_m^1 = \mathbb{Z}_2$  ,  $B = \text{---}$

2 P $\bar{1}$   (many)

Monoclinic  $G_m = C_2$  ;  $G_m^1 = \mathbb{Z}_2$  ,  $B = \text{---}$   
(  $G_m^1 = \Delta_2$  ,  $B = \text{---}, \text{---}, \text{---}$  )

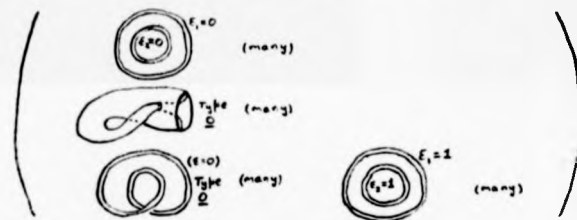
3 P2



4 P2<sub>1</sub>



5 C2



Monoclinic  $G_m = \bar{C}_2$  ;  $G_m^1 = \{id\}$  ,  $B = \text{---}$   
(  $G_m^1 = \Delta_2$  ,  $B = \text{---}, \text{---}, \text{---}$  )

6 Pm



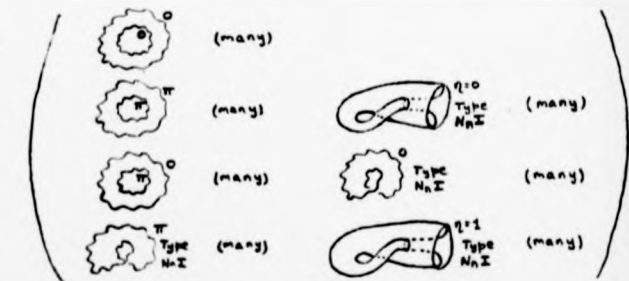
7 Pc



8 Cm



9 Cc



Monoclinic  $G_m = \bar{C}_2$  ;  $G_m^1 = \mathbb{Z}_2$  ,  $B = \text{---}$   
(  $G_m^1 = \Delta_2$  ,  $B = \text{---}, \text{---}, \text{---}$  )

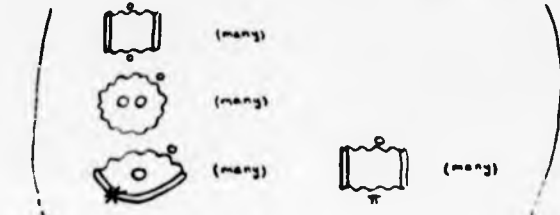
10 P2/m



11 P2<sub>1</sub>/m



12 C2/m

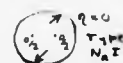




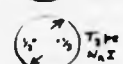
31  $Pm\bar{2}_1$



32  $P6_32$



33  $Pna2_1$



34  $Pnn2$



35  $Cmn2$



36  $Cmc2_1$



37  $Ccc2$



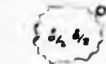
38  $Amn2$



39  $Abm2$



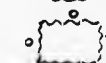
40  $Ama2$



41  $Abn2$



42  $Fmm2$



43  $Fdd2$



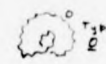
44  $Imm2$



45  $Iba2$



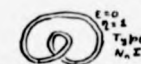
46  $Ima2$



and twice



and twice



and twice



and twice



and twice



and twice



and twice



and twice



Orthorhombic  $G_0 = \bar{D}_2$  ;

$G_0^1 = \Delta_0$  ,  $B = \begin{matrix} \square \\ \square \end{matrix}$  ,  $\begin{matrix} \square \\ \square \end{matrix}$  ,  $\begin{matrix} \square \\ \square \end{matrix}$  ,  $\begin{matrix} \square \\ \square \end{matrix}$

47  $Fmmm$



three times

48  $Pnnn$



three times

49  $Pccm$



and twice



50  $Pban$



and twice



51 Pmma



52 Pnna



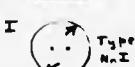
53 Pmna



54 Pcca



55 Pbca



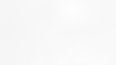
and twice



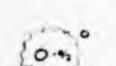
56 Pcen



and twice



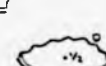
57 Pbcm



58 Pnmm



and twice



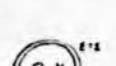
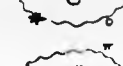
59 Pmmn



and twice



60 Pbcn



61 Pbca



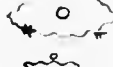
three times



62 Pnma



63 Cmcn



64 Cmca



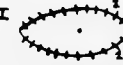
65 Cmmm



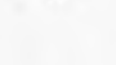
and twice



66 Cccn



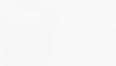
and twice



67 Cmma



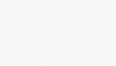
and twice



68 Ccca



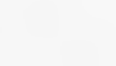
and twice



69 Fmmm



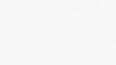
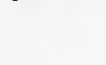
three times



70 Fddd



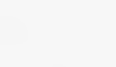
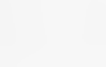
three times



71 Immm



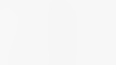
three times



72 Ibam



and twice



73 Iba



three times

74 Imma



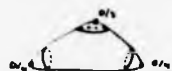
and twice



Tetragonal

$$G_0 = C_4 ; G_0^* = \mathbb{Z}_4 , B = \text{triangle}$$

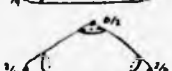
75 P4



76 P4<sub>1</sub>



77 P4<sub>2</sub>



78 P4<sub>3</sub>



79 I4



80 I4<sub>1</sub>



Tetragonal

$$G_0 = \tilde{C}_4 ; G_0^* = \mathbb{Z}_4 , B = \text{triangle}$$

81 P4̄



82 I4̄



Tetragonal

$$G_0 = \hat{C}_4 ; G_0^* = \mathbb{Z}_4 , B = \text{triangle}$$

83 P4/m



84 P4<sub>2</sub>/m



85 P4/n



86 P4<sub>3</sub>/n



87 I4/m



88 I4<sub>3</sub>/a



Tetragonal

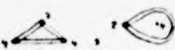
$G_m = D_8$

:

$G_m^b = \Delta_4$

,

B =



89  $P4_22$



90  $P4_22$



91  $P4_22$



92  $P4_22$



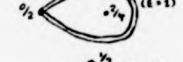
93  $P4_22$



94  $P4_22$



95  $P4_22$



96  $P4_22$



97  $I4_22$



98  $I4_22$



Tetragonal

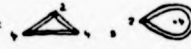
$G_m = \bar{C}_8$

:

$G_m^b = \Delta_4$

,

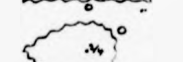
B =



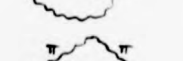
99  $P4mm$



100  $P4mm$



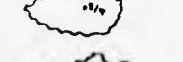
101  $P4_2cm$



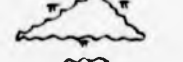
102  $P4_2cm$



103  $P4_2cm$



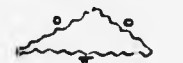
104  $P4_2cm$



105  $P4_2cm$



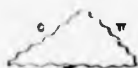
106  $P4_2cm$



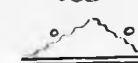
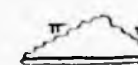
107  $I4mm$



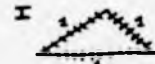
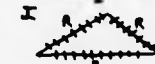


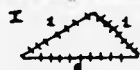
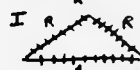
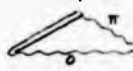
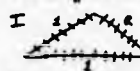
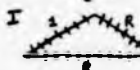
108  $I4cm$ 109  $I4_1cd$ 110  $I4_2cd$ 

Tetragonal  $G_m = \bar{D}_2$  ;  $G_m^h = \Delta$  ,  $B = \triangle$  ,  $\odot$

111  $P4_2m$ 112  $P4_2c$ 113  $P4_2nm$ 114  $P4_2_1c$ 115  $P4m2$ 116  $P4c2$ 117  $P4_12_1$ 118  $P4_22_1$ 119  $I4m2$ 120  $I4_22$ 121  $I4_22m$ 122  $I4_22d$ 

Tetragonal  $G_m = \bar{D}_{16}$  ;  $G_m^h = \Delta_8$  ,  $B = \triangle$  ,  $\odot$

123  $P4_1/mmm$ 124  $P4_2/mcc$ 125  $P4_1/abm$ 126  $P4_1/nnc$ 

127  $P4/m\bar{3}m$ 128  $P4/mnc$ 129  $P4/nmm$ 130  $P4/mcc$ 131  $P4_2/mnc$ 132  $P4_2/mcm$ 133  $P4_2/m\bar{3}c$ 134  $P4_2/nmm$ 135  $P4_2/m\bar{3}c$ 136  $P4_2/nmm$ 137  $P4_2/mnc$ 138  $P4_2/mcm$ 139  $I4/mmm$ 140  $I4/mcm$ 141  $I4_2/amd$ 142  $I4_2/acd$ 

Hexagonal  $G_m = C_3$  ;  $G'_m = Z_3$  ,  $B = \begin{matrix} \text{---} \\ \text{---} \end{matrix}$

143  $P3$ 144  $P3_1$ 145  $P3_2$ 146  $R3$ 

Hexagonal

$$G_0 = \bar{C}_6$$

$$G_0^1 = \mathbb{Z}_6$$

$$B = \triangle$$

147  $\bar{P}3$



148  $R\bar{3}$



Hexagonal

$$G_0 = D_6$$

$$G_0^1 = \Delta_6$$

$$B = \triangle, \triangle, \triangle$$

149  $P312$



150  $P321$



151  $P3_112$



152  $P3_21$



153  $P3_112$



154  $P3_21$



155  $R32$



Hexagonal

$$G_0 = \bar{C}_6$$

$$G_0^1 = \Delta_6$$

$$B = \triangle, \triangle, \triangle$$

156  $P3m1$



157  $P31m$



158  $P3c1$



159  $P31c$



160  $R3m$



161  $R3c$



Hexagonal

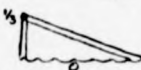
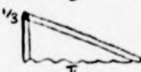
$$G_0 = \bar{D}_{12}$$

$$G_0^1 = \Delta_{12}$$

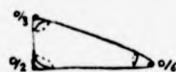
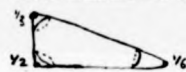
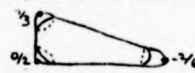
$$B = \triangle$$

162  $P\bar{3}1m$



163  $P\bar{3}1c$ 164  $P\bar{3}m1$ 165  $P\bar{3}c1$ 166  $R\bar{3}m$ 167  $R\bar{3}c$ 

Hexagonal  $G_m = C_6$  ;  $G_m^b = Z_6$  ,  $B =$

168  $P6$ 169  $P6_1$ 170  $P6_5$ 171  $P6_2$ 172  $P6_4$ 173  $P6_3$ 

Hexagonal  $G_m = \hat{C}_6$  ;  $G_m^b = Z_3$  ,  $B =$

174  $P\bar{6}$ 

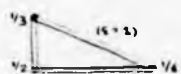
Hexagonal  $G_m = \hat{C}_{12}$  ;  $G_m^b = Z_6$  ,  $B =$

175  $P6/m$ 176  $P6_3/m$ 

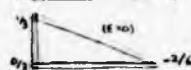
Hexagonal  $G_m = D_{12}$  ;  $G_m^b = \Delta_{12}$  ,  $B =$

177  $P6_{22}$ 178  $P6_3_{22}$ 

179  $P6_322$



180  $P6_322$



181  $P6_322$



182  $P6_322$



Hexagonal

$$G_{\infty} = \bar{C}_{12} ; G^k_{\infty} = \Delta_{12} , B = \begin{array}{c} \nearrow \\ \searrow \end{array}$$

183  $P6mm$



184  $P6cc$



185  $P6_3cm$



186  $P6_3mc$



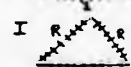
Hexagonal

$$G_{\infty} = \bar{D}_{12} ; G^k_{\infty} = \Delta_{12} , B = \begin{array}{c} \nearrow \\ \searrow \end{array}$$

187  $P\bar{6}m2$



188  $P6c2$



189  $P\bar{6}2m$



190  $P\bar{6}2c$



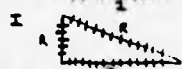
Hexagonal

$$G_{\infty} = \bar{D}_{24} ; G^k_{\infty} = \Delta_{12} , B = \begin{array}{c} \nearrow \\ \searrow \end{array}$$

191  $P6/mmm$



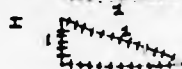
192  $P6/mcc$



193  $P6_3/mcm$



194  $P6_3/mnc$



graphic space group each crystal corresponds; the aim of Table 4 is to redress this. It lists each of the first 194 space groups in the usual order (usual to crystallographers, that is), denoting each group by its *International Number* and the widely-used *International Symbol*, together with all the  $S^1$ -fibred 3-crystals which correspond to it. The list is divided into the 27 *crystal classes* familiar to crystallographers.

The table offers extra information over that <sup>which</sup>  $\lambda$  was previously available, since it links the two methods of classification: the crystallographic, looking at non-conjugate group-actions on  $\mathbb{R}^3$ ; and the topological, looking at non-isomorphic quotient spaces  $\mathbb{R}^3/G$ . The connection between the two is established using diagrams like that of (0.2.4) to perform a case-by-case analysis of the group actions and see which fibred 3-crystals they yield. It therefore serves as a cross-check of the work in [B + S].

(2.5.1) The table is divided into three columns:

Column 1 specifies the *International Number* (1-194), as used in the *International Tables for X-Ray Crystallography* [ITC]. The remaining 36 groups (195-230) are not listed since they fail to preserve any direction in  $\mathbb{R}^3$  and therefore do not produce  $S^1$ -fibred 3-crystals. Their (non-fibred) 3-crystals are dealt with in §6.

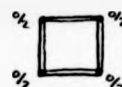
Column 2 gives the corresponding *International Symbol* for each of the groups. This, along with the *Schönflies symbol*, is a universally recognized means by which crystallographers denote crystallographic groups. Tables linking the two types of notation can be found in, for instance, [ITC] or [B + C].

Column 3 lists all the  $S^1$ -fibred 3-orbifolds corresponding to each group. Although it may preserve infinitely many distinct directions in  $\mathbb{R}^3$ , for any given group there are at most three corresponding  $S^1$ -fibred 3-orbifolds. This is because fibering in these diverse directions leads to objects which are, up to base-free fibred isomorphism, the same as one another: this is best seen by remembering that a base-free fibred isomorphism down in the orbifold, when it is geodesically fibred, is the same as an affine homeomorphism up in the covering space  $\mathbb{R}^3$ . Whenever more than one choice of direction in  $\mathbb{R}^3$  leads to fibred-isomorphic objects this is noted e.g.



(many)

,



(three times)

The standard tables for these groups usually describe them by means of the *Seitz notation*, which specifies generators for the group action on  $\mathbb{R}^3$  with respect to a standard right-hand set of axes  $(O_x, O_y, O_z)$ . When there is more than one fibred 3-orbifold for a particular group the first one listed corresponds to fibering  $\mathbb{R}^3$  in the  $O_z$  direction, with the group specified by the Seitz symbols as given in [ITC]. Conveniently, the symbols there are chosen so that the  $O_z$  direction is always a preserved one, when such exists. Crystallographers will recognize this as the *principal axis*.

Choices of distinct fibrations occur only in certain cases:

The *triclino* groups (Nos. 1 & 2) preserve any "rational" direction in  $\mathbb{R}^3$  - all of them lead to the same fibred crystal. Fibering in an irrational direction (so a fibre passes through only one lattice point) leads to a line bundle.

For the *monoclinic* groups (3-15), there exists one direction leading to a unique crystal. The alternative crystals occur by choosing rational fibrations in the plane perpendicular to this (0.7.2).

The *orthorhombic* groups (16-74) preserve exactly three mutually perpendicular directions, leading to three fibred orbifolds which in general need not be equivalent.

The remaining *tetragonal* and *hexagonal* systems of groups <sup>(75-142) and (143-174)</sup> preserve exactly one direction in  $\mathbb{R}^3$ , with a correspondingly unique fibred crystal Q.

The *cubic* system of groups (195-230) fails to preserve any direction and is omitted here: it is dealt with in §6.

The table is divided into the above systems of groups and subdivided into *geometric crystal classes* corresponding to the 27 distinct point groups  $G_\infty$  which occur (0.2.1). In each of these 27 cases, the restriction of the action of  $G_\infty$  to the base space is given, denoted by  $G_\infty^b$ . It determines what possibilities for B are available, and these are drawn next to it.

(2.5.2) Orientation: As has already been mentioned, the group-actions are specified via the Seitz notation, with reference to a right-hand set of axes. Rotations about  $O_z$ , say, are always anti-clockwise *looking from above* (i.e. from z to 0). Thus, for the group  $P4_1$ , which has Seitz symbol  $\{C_{4z}^+ : 00\frac{1}{4}\}$ , meaning "rotate through  $2\pi/4$  anticlockwise about the  $O_z$ -axis, simultaneously moving along the  $O_z$ -axis  $\frac{1}{4}$  of a unit in the positive direction", the fibration looks like:

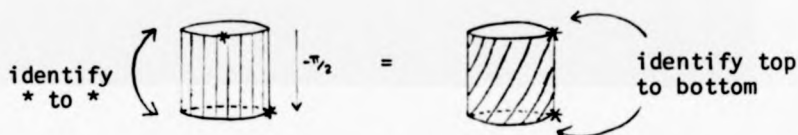




Using Bonahon and Siebenmann's orientation conventions for orbifolds, to which we adhere throughout, a cone point with slope  $\beta/\alpha$  represents  $(D^2 \times S^1)/\mathbb{Z}_\alpha$  where  $\mathbb{Z}_\alpha$  acts by :

$$\begin{cases} \text{rotate } D^2 & \text{by } 2\pi/\alpha & \text{(anticlockwise)} \\ \text{rotate } S^1 & \text{by } -2\pi\beta/\alpha & \text{(where "+" indicates "upwards")} \end{cases}$$

so  $\frac{1}{4}$  looks like:



The unfortunate consequence, from the point of view of presentation, is that

$$\mathbb{R}^3/P4_1 \cong \text{cone with angles } \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \text{ and } \frac{\pi}{2}, \text{ whereas } \mathbb{R}^3/P4_3 \cong \text{cone with angles } \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \text{ and } \frac{\pi}{2}.$$

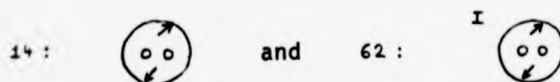
This pattern is repeated for all the other amphicheiral pairs.

(2.5.3) Notation: All the 2-orbifolds  $B$  can be recognized from Table 2, so the orders of their singular points have been omitted from the type diagrams to reduce clutter.

As previously stated, the convention is followed that all slopes lie in the range  $-\frac{1}{2} < \beta/\alpha \leq \frac{1}{2}$ , further restricted to  $0 \leq \beta/\alpha \leq \frac{1}{2}$  wherever possible - namely when the weak data function  $\psi$  does not assign Seifert type  $\underline{0}$  to  $B$ .

When a parity for a Montesinos cycle is given in parentheses this is because it is completely determined by other factors - this is the case whenever there is only one such cycle and  $Q$  is orientable. When the  $\epsilon = 0$  and  $\epsilon = 1$  cases are equivalent, as occurs in the third picture for group 52 (Pnna) and the pictures for groups 70 (Fddd) and 122 (I42d), it is omitted altogether. In each of these cases the  $\epsilon = 0$  and  $\epsilon = 1$  diagrams are equivalent by move  $(d_1)$  followed by move  $(b_2)$  (see (2.1.8)). Whenever there is more than one Montesinos cycle, parities are labelled  $\epsilon_1, \epsilon_2$  etc. We also assume for any such cycle  $C_i$  that  $\epsilon'(C_i) = \epsilon(C_i)$  - this can always be made so using move (c) of (2.1.8), provided that  $n$  suffers the corresponding change.

The assignment of Seifert type 0 has been omitted wherever it is the only type possible - this avoids unnecessary tedium with the orientable groups. Thus, wherever appropriate, if no Seifert type is given then  $p^{-1}(\text{int } |B|)$  is an orientable topological manifold. As a consequence the only diagrams which are labelled with a Seifert type are those for which  $|B| = \text{torus, Mobius band, Klein bottle or projective plane}$ . In fact all such objects realize more than one Seifert type except for



Crystals which are mI-fibred are denoted by an "I" on their left.

For group 33 (Pna  $2_1$ ) and the first picture for 43 (Fdd2), the cases  $n = 0$  and  $n = 1$  are equivalent by move  $(d_2)$  followed by move  $(b_1)$ , so  $n$  has been omitted.

### §3. Black and White Crystallographic Groups.

The novel approach to crystallographic groups of investigating the orbifold structure of their compact quotients can be extended to an examination of black and white groups [Schw 1,2],  $[J + R]$ ,  $[B + C]$ .

(3.0) Definition: A *black and white group* consists of a euclidean crystallographic space group  $G$  and an index two subgroup  $H \leq G$ , written as an ordered pair  $(G, H)$ .

Two such pairs  $(G, H)$  and  $(G', H')$  are said to be *equivalent* if there is an orientation-preserving affine homeomorphism  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $G' = fGf^{-1}$  and  $H' = fHf^{-1}$ . //

(3.1) The history of this branch of crystallography is best considered elsewhere; a good brief survey can be found in the aforementioned [Schw 2], which also boasts a large and satisfyingly obscure bibliography. However, it will do no harm for us to sketch some of the background here.

Much of the early work was by Shubnikov [Shub.], for which reason black and white groups are also called *Shubnikov groups*. Given an  $n$ -dimensional space group  $H$ , a new group  $M$  can be created by allowing an additional operation of "antisymmetry". If this is represented by  $\theta$ , with  $\theta^2 = \text{identity}$ , then we have

$$M = H \cup \theta H,$$

where  $H$  and  $G = H \cup H'$  are both space groups and  $|G:H| = 2$ .

If this seems a little ethereal then a few moments thought about how to represent such a group in practice should bring us back down to earth. As we saw in (0.2.4), a convenient means of portraying a space group is to consider a suitable motif at the origin and see how and where it is transformed under the action of the group. We can do this by bringing into the picture an additional feature which is allowed to have two states; the obvious choice is to allow the motif to be transferred from black to white (and vice-versa) under the action of  $\theta$ .

(3.1.1) Example:



This method of presenting black and white groups can obviously be extended to the 3-dimensional case, but with a corresponding increase in complexity which makes such diagrams very cumbersome.

(3.1.2) A black and white group  $(G, H)$  determines exact sequences

$$\begin{aligned}
 0 &\rightarrow G_0 \rightarrow G \rightarrow G_\infty \rightarrow 1 \\
 &\quad \cup \quad \cup \quad \cup \\
 0 &\rightarrow H_0 \rightarrow H \rightarrow H_\infty \rightarrow 1
 \end{aligned}$$

(in the notation of (0.2.1)). As an immediate consequence we can subdivide the Shubnikov groups  $M$  into two types:

(3.1.3) Type III Shubnikov Space Groups.

These have  $G_0 \cong H_0$  and  $|G_\infty : H_\infty| = 2$ . Thus

$$M \cong H \cup \theta(G-H),$$

where  $H' = G-H$  contains no pure translations.

(3.1.4) Type IV Shubnikov Space Groups.

These have  $G_\infty \cong H_\infty$  and  $|G_0 : H_0| = 2$ . Thus

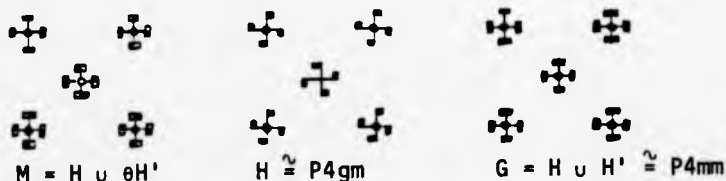
$$M \cong H \cup \theta\{E|t_g\}H,$$

where  $\{E|t_g\}$  is the Seitz space-group notation for a (non-trivial) pure translation  $t_g$ ;  $E$  just represents the identity element of the point group. In this instance we have

$$H' = G-H = \{E|t_g\}H.$$

An example of a type III group is provided by (3.1.1) above, for which the extra (white) operations are clearly elements of  $G_\infty$ . The following is an example of a type IV group:

(3.1.5) Example:



Although at first it may seem that the point group  $G_\infty$  has again been enlarged (as in the previous example), closer examination shows us that

G is obtained from H by allowing a translation to the centre of the four lattice points which are marked for H, changing the lattice from "primitive" to "body centred". //

The alert reader will have noticed that nothing has been said about type I or type II Shubnikov groups. The former are simply the crystallographic space groups defined in (0.2); the latter are the "grey" groups, and are trivially obtained as

$$M \cong H \cup \theta H$$

for any space group H. Neither type will be referred to again. In fact, since the distinction between type III and type IV groups will not generally be of concern to us, we will henceforth refer just to black and white groups.

Aside from mathematical curiosity, why should anyone be interested in black and white groups? One reason is that they represent genuine physical objects. Whilst classical space groups depict crystalline structures, black and white groups have the added advantage of representing crystals which exhibit additional properties such as ferromagnetism and ferroelectricity - see [Schw.2], [Cr.]. There is no need for us to restrict ourselves to two colours: the theory of n-coloured crystal symmetry is gradually being developed and already finding applications - see [Schw.2].

A more pertinent reason for investigating black and white groups, from the mathematicians point of view at least, is that the antisymmetry operation can be thought of as acting on a fourth co-ordinate s which

is allowed to take two values, so that

$$\theta(x,y,z,s) = (x,y,z,-s) .$$

This affords a useful step up from 3-dimensional space groups to their 4-dimensional counterparts. Although 4-dimensional space groups have recently been classified [Bro], they are far from being well understood. Handling black and white groups from the topological side of things as we have attempted to do here seems potentially to provide a neat way of looking at the corresponding 4-dimensional orbifolds by allowing us, in many cases, to understand them in terms of a double covering of  $S^1$ -fibred 3-orbifolds.

(3.2) Returning to Definition (3.0) the first things to note are that if  $(G,H)$  is a black and white group then  $H$  is automatically a crystallographic space group and that if  $G$  preserves a direction  $V$  in  $\mathbb{R}^3$  then so does  $H$ . Thus if  $G$  yields an  $S^1$ -fibred 3-crystal

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B$$

then

$$(\mathbb{R}^3, V)/H = Q' \xrightarrow{p'} B'$$

is a fibred double cover of  $Q \xrightarrow{p} B$  i.e. we have the following commutative diagram:

$$(3.2.1) \quad \begin{array}{ccccc} (\mathbb{R}^3, V)/H & = & Q' & \xrightarrow{\phi} & Q & = & (\mathbb{R}^3, V)/G \\ & & p' \downarrow & & \downarrow p & & \\ & & B' & \xrightarrow{\bar{\phi}} & B & & \end{array}$$

$\phi:Q' \rightarrow Q$  is the fibred double covering map and  $\bar{\phi}:B' \rightarrow B$ , its projection to the base spaces, is either a homeomorphism or a double covering map of 2-crystals.

It is tempting to assume that if a group  $G$  gives us a fibred 3-crystal then the black and white groups  $(G, G_1)$  correspond to fibred double covers of it up to some suitable equivalence. Unfortunately life is not quite that simple: some groups preserve more than one direction in  $\mathbb{R}^3$ , which can lead to complications. These are dealt with in (3.6.1) - (3.6.3).

From now on, unless otherwise stated, a "group" refers to a "euclidean crystallographic space group". This will continue to apply in the (mainly topological) §§4 and 5. When the action of such a group  $G$  is orientation-preserving then  $Q = \mathbb{R}^3/G$  is orientable. We shall assume that  $\mathbb{R}^3$  has the standard orientation by a right-hand set of axes, projecting to the same for  $Q$ . It will be implicit that an isomorphism of such orbifolds is required to be orientation-preserving.

(3.3) Let  $G$  be a group which preserves some direction  $V$  in  $\mathbb{R}^3$  giving a fibred quotient

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B.$$

We wish to classify the black and white pairs  $(G, H)$  up to equivalence. First a notion of equivalence for fibred double covers of  $Q \xrightarrow{p} B$  is needed.



(3.3.1) Definition: Let  $Q_1' \xrightarrow{\phi_1} Q$  and  $Q_2' \xrightarrow{\phi_2} Q$  be two

$$\begin{array}{ccc} Q_1' & \xrightarrow{\phi_1} & Q \\ p_1' \downarrow & & \downarrow p \\ B_1' & \xrightarrow{\phi_1} & B \end{array} \quad \begin{array}{ccc} Q_2' & \xrightarrow{\phi_2} & Q \\ p_2' \downarrow & & \downarrow p \\ B_2' & \xrightarrow{\phi_2} & B \end{array}$$

fibred double covers of an  $S^1$ -fibred 3-orbifold  $Q \xrightarrow{p} B$ . Let  $\rho_1: Q_1' \rightarrow Q_1'$  and  $\rho_2: Q_2' \rightarrow Q_2'$  be the respective fibred covering translations, which must be orientation-preserving when  $Q$  is orientable. The covers are said to be *equivalent* if there exists a fibred isomorphism  $\psi: Q_1' \rightarrow Q_2'$  such that the following diagram commutes:

$$\begin{array}{ccc} Q_1' & \xrightarrow{\psi} & Q_2' \\ p_1' \downarrow & & \downarrow p_2' \\ B_1' & \xrightarrow{\bar{\psi}} & B_2' \end{array}$$

(where  $\bar{\psi}: B_1' \rightarrow B_2'$  is just the projection of  $\psi$  to the base orbifolds) and such that

$$\psi \circ \rho_1 \circ \psi^{-1} = \rho_2$$

i.e. the fibred isomorphism between  $Q_1' \xrightarrow{p_1'} B_1'$  and  $Q_2' \xrightarrow{p_2'} B_2'$  projects to a fibred automorphism of  $Q \xrightarrow{p} B$ . //

Proposition  
(3.3.2)  $\lambda$  Let  $G$  be a group which preserves some direction  $V$  in  $\mathbb{R}^3$ . Let  $(G, H_1)$  and  $(G, H_2)$  be black and white groups.

$$\begin{aligned} \text{If } (\mathbb{R}^3, V)/H_1 &= Q_1' \xrightarrow{p_1'} B_1' \\ \text{and } (\mathbb{R}^3, V)/H_2 &= Q_2' \xrightarrow{p_2'} B_2' \end{aligned}$$

are equivalent fibred double covers of

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B ,$$

then  $(G, H_1)$  and  $(G, H_2)$  are equivalent black and white pairs.

Proof: We have a fibred isomorphism  $\psi': Q'_1 \rightarrow Q'_2$  such that

$$\begin{array}{ccc} Q'_1 & \xrightarrow{\psi'} & Q'_2 \\ p'_1 \downarrow & & \downarrow p'_2 \\ B'_1 & \xrightarrow{\bar{\psi}'} & B'_2 \end{array} \quad \text{commutes, and } \psi' \circ \rho_1 \circ \psi'^{-1} = \rho_2 ,$$

where  $\rho_1: Q'_1 \rightarrow Q_1$  and  $\rho_2: Q'_2 \rightarrow Q_2$  are the fibred covering translations. This projects to a well-defined fibred automorphism  $\psi: Q \rightarrow Q$ ; note that  $\psi$  is orientation-preserving when  $Q$  is oriented, since this implies that all the group-actions in  $G$  act orientation-preservingly.

Induced by  $\psi: Q \rightarrow Q$  is a group automorphism

$$\psi_* : G \xrightarrow{\cong} G \quad (G \cong \pi_1(Q)) .$$

Since  $Q'_1 \xrightarrow{\psi'} Q'_2$  commutes, we have  $\psi_*|_{H_1} = \psi'_*$

$$\begin{array}{ccc} Q'_1 & \xrightarrow{\psi'} & Q'_2 \\ \downarrow / \rho_1 & & \downarrow / \rho_2 \\ Q & \xrightarrow{\psi} & Q \end{array} \quad \text{i.e. } \psi_*(H_1) = H_2 .$$

Bieberbach's second theorem (0.6.4) gives us an affine homeomorphism

$\tilde{f}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that

$$\psi_*(g) = \tilde{f}g\tilde{f}^{-1} \quad \text{for all } g \in G$$

$$\text{i.e. } G = \tilde{f}G\tilde{f}^{-1} \quad \text{and} \quad H_2 = \tilde{f}H_1\tilde{f}^{-1}, \quad \text{as required.}$$

Finally, note that  $\tilde{f}$  is orientation-preserving when  $\psi$  is.  $\square$

Now it would be nice to prove something in the opposite direction. Sadly, it is not true in general that equivalent black and white groups give equivalent fibred double covers.

(3.4) Definition: Given a crystallographic space group  $G$  and a direction  $V$  in  $\mathbb{R}^3$  which is preserved by  $G$ , we say that

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{P} B$$

is *unique* if, for any  $W \neq V$  which is also preserved by  $G$ ,

$$(\mathbb{R}^3, W)/G = Q' \xrightarrow{P'} B'$$

is not fibred-isomorphic to  $Q \xrightarrow{P} B$ .

//

(3.4.1) Lemma: Given a unique fibration

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{P} B,$$

if  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is an affine homeomorphism such that  $fGf^{-1} = G$ , then  $f$  must preserve the direction  $V$  i.e.

$$f(\mathbb{R}^3/V) = \mathbb{R}^3/V.$$

Proof:

$$\begin{aligned}
 \mathbb{R}^3/f(V) &= f(\mathbb{R}^3/V) && \text{since } f \text{ is affine} \\
 &= fG(\mathbb{R}^3/V) && \text{since } G \text{ preserves } V \\
 &= Gf(\mathbb{R}^3/V) \\
 &= G(\mathbb{R}^3/f(V)) .
 \end{aligned}$$

Thus  $G$  preserves  $\mathbb{R}^3/f(V)$ , so we can look at

$$(\mathbb{R}^3, f(V))/G = Q' \xrightarrow{p'} B' .$$

We know that  $Q \xrightarrow{p} B$  is unique to  $V$ , so if it can be shown that  $Q' \xrightarrow{p'} B'$  is fibred-isomorphic to  $Q \xrightarrow{p} B$  then we must have  $f(\mathbb{R}^3/V) = \mathbb{R}^3/V$  as required.

But

$$\begin{aligned}
 (\mathbb{R}^3, f(V))/G &= (\mathbb{R}^3, f^{-1} \circ f(V))/f^{-1}Gf \\
 &= (\mathbb{R}^3, V)/G
 \end{aligned}$$

i.e. the two fibred orbifolds are the same, so  $f(\mathbb{R}^3/V) = \mathbb{R}^3/V$ .  $\square$

(3.5) Proposition: Given a group  $G$  preserving a direction  $V$  in  $\mathbb{R}^3$  such that

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B$$

is unique, if  $(G, H_1)$  and  $(G, H_2)$  are equivalent black and white groups then

$$(\mathbb{R}^3, V)/H_1 = Q'_1 \xrightarrow{p'_1} B'_1$$

and

$$(\mathbb{R}^3, V)/H_2 = Q'_2 \xrightarrow{p'_2} B'_2$$

are equivalent as fibred double covers of  $Q \xrightarrow{p} B$ .

Proof: We have  $(G, H_1)$  and  $(G, H_2)$  and an affine homeomorphism

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad \text{such that}$$

$$fGf^{-1} = G \quad \text{and} \quad fH_1f^{-1} = H_2.$$

The Lemma tells us that  $f(\mathbb{R}^3/V) = \mathbb{R}^3/V$ . We wish to show that there is a fibred isomorphism  $\psi: Q_1^i \rightarrow Q_2^i$  such that the following diagram commutes:

$$\begin{array}{ccc} (\mathbb{R}^3, V)/H_1 & = & Q_1^i \xrightarrow{\psi} Q_2^i = (\mathbb{R}^3, V)/H_2 \\ p_1^i \downarrow & & \downarrow p_2^i \\ B_1^i & \xrightarrow{\bar{\psi}} & B_2^i \end{array}$$

and

$$\psi \circ p_1 \circ \psi^{-1} = p_2.$$

where  $\rho_1: Q_1^i \rightarrow Q_1^i$  and  $\rho_2: Q_2^i \rightarrow Q_2^i$  are the respective fibred covering translations of  $Q_1^i$  and  $Q_2^i$  as double covers of  $Q \xrightarrow{p} B$ , and  $\bar{\psi}: B_1^i \rightarrow B_2^i$  is the projection of  $\psi$  to the base spaces.

Let  $x \in Q_1^i$ , so  $x = H_1 \tilde{x}$  for some  $\tilde{x} \in \mathbb{R}^3$ . Then

$$\begin{aligned} fH_1 \tilde{x} &= H_2 f\tilde{x} = H_2 \tilde{y}, \quad \text{where } \tilde{y} = f(\tilde{x}) \in \mathbb{R}^3 \\ &= y \in Q_2^i. \end{aligned}$$

By defining  $\psi(x) = y$  we have a well-defined map  $\psi: Q_1^i \rightarrow Q_2^i$  with the following properties:

(3.5.1)  $\psi$  is fibre-preserving: Let  $x, y \in Q_1^i$  lie on the same fibre.

Lift them to  $\tilde{x}, \tilde{y} \in \mathbb{R}^3$  (i.e.  $x = H_1 \tilde{x}$ ,  $y = H_1 \tilde{y}$ ). We are free to choose  $\tilde{x}$  and  $\tilde{y}$  so that they lie on the same 1-dimensional affine subspace preserved by  $G$ ; but then so do  $f(\tilde{x})$  and  $f(\tilde{y})$ . Thus  $\psi(x) = H_2 f(\tilde{x})$  and  $\psi(y) = H_2 f(\tilde{y})$  lie on the same fibre in  $(\mathbb{R}^3, V)/H_2 = Q_2' \xrightarrow{p_1'} B_2'$ .  $\square_1$

(3.5.2)  $\psi$  is injective: Let  $x, y \in Q_1'$  with  $x \neq y$ .  $x = H_1 \tilde{x}$  and  $y = H_1 \tilde{y}$  for some  $\tilde{x}, \tilde{y} \in \mathbb{R}^3$ , and so we have

$$H_1 \tilde{x} \cap H_1 \tilde{y} = \emptyset, \text{ since } x \neq y.$$

But  $fH_1 \tilde{x} = H_1 f\tilde{x}$  and  $fH_1 \tilde{y} = H_1 f\tilde{y}$ ; since  $f$  is a homeomorphism we must have

$$H_2 f\tilde{x} \cap H_2 f\tilde{y} = \emptyset$$

$$\text{i.e. } \psi(x) \neq \psi(y).$$

$\square_2$

(3.5.3)  $\psi$  is surjective: Let  $y \in Q_2'$ , so  $y = H_2 \tilde{y}$  for some  $\tilde{y} \in \mathbb{R}^3$ . Then

$$f^{-1}H_2 \tilde{y} = H_1 f^{-1} \tilde{y}.$$

$$\text{Hence } \psi^{-1}(y) = H_1(f^{-1} \tilde{y}) \in \mathbb{R}^3/H_1 = Q_1'.$$

$\square_3$

Thus  $\psi: Q_1' \rightarrow Q_2'$  is a bijection; it is clearly also both ways continuous.

(3.5.4)  $\psi$  makes the atlases compatible: Let  $U = \tilde{U}/\Gamma$  be a chart in

$Q_1^i$ . Then the pre-image of  $U$  in  $\mathbb{R}^3$  consists of disjoint copies  $\tilde{U}_i$  of  $\tilde{U}$  (indexed by  $i \in I$ ), and

$$\left( \bigcup_{i \in I} \tilde{U}_i \right) / H_1 \cong \tilde{U} / \Gamma = U.$$

Thus 
$$\begin{aligned} \psi(U) &= (f(\bigcup_{i \in I} \tilde{U}_i)) / H_2 \\ &= \left( \bigcup_{i \in I} (f\tilde{U}_i) \right) / fH_1 f^{-1} \cong \left( \bigcup_{i \in I} \tilde{U}_i \right) / H_1. \end{aligned}$$

□<sub>4</sub>

(3.5.5)  $\psi \circ \rho_1 = \rho_2 \circ \psi$  : Let us look at how  $\psi \circ \rho_1$  and  $\rho_2 \circ \psi$  are defined:

$$\begin{array}{ccccccc} \psi \circ \rho_1 : \mathbb{R}^3 \supseteq H_1 \tilde{x} & \xrightarrow{\gamma_1 \in G-H_1} & \gamma_1 \tilde{x} \in H_1(\gamma_1 \tilde{x}) & \xrightarrow{f} & f\gamma_1 \tilde{x} \in (fH_1)(\gamma_1 \tilde{x}) = (H_2 f)(\gamma_1 \tilde{x}) \\ \downarrow / H_1 & \downarrow & \downarrow & & \downarrow \\ \mathbb{R}^3 / H_1 = Q_1^i \ni x & \xrightarrow{\rho_1} & \rho_1(x) & \xrightarrow{\psi} & \psi \circ \rho_1(x) \end{array}$$

$$\begin{array}{ccccccc} \rho_2 \circ \psi : \mathbb{R}^3 \supseteq H_1 \tilde{x} & \xrightarrow{f} & f\tilde{x} \in fH_1 \tilde{x} = H_2 f\tilde{x} & \xrightarrow{\gamma_2 \in G-H_2} & \gamma_2 f\tilde{x} \in H_2(\gamma_2 f\tilde{x}) \\ \downarrow / H_1 & \downarrow & \downarrow & & \downarrow \\ \mathbb{R}^3 / H_1 = Q_1^i \ni x & \xrightarrow{\psi} & \psi(x) & \xrightarrow{\rho_2} & \rho_2 \circ \psi(x) \end{array}$$

We must show that the sets

$$H_2 f \gamma_1 \tilde{x} \quad \text{and} \quad H_2 \gamma_2 f \tilde{x}$$

are equal. Since  $G = fGf^{-1}$  and  $H_2 = fH_1 f^{-1}$ , we must have

$$G - H_2 = f(G-H_1)f^{-1}$$

and hence

$$G - H_1 = f^{-1}(G-H_2)f .$$

Thus, since  $\gamma_1 \in (G-H_1)$ , we have

$$\gamma_1 = f^{-1}\gamma_2 f \quad \text{for some } \gamma_2 \in G-H_2 .$$

$$\begin{aligned} \text{and so } H_2 f \gamma_1 \tilde{x} &= H_2 f (f^{-1} \gamma_2 f) \tilde{x} \\ &= H_2 \gamma_2 f \tilde{x} \quad \text{for some (and hence all) } \gamma_2 \in (G-H_2) . \end{aligned}$$

□<sub>5</sub>

In conclusion note that if  $Q_1^+$  (and hence  $Q_2^+$ ) is orientable then  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is orientation-preserving and so  $\psi: Q_1^+ \rightarrow Q_2^+$  is orientation-preserving. □

(3.6) Since most of the 230 space groups  $G$  - 180 of them, in fact - preserve some direction  $V$  in  $\mathbb{R}^3$  leading to a unique fibration (in the sense of Definition 3.4), Proposition (3.5) is sufficient to allow us to classify all the black and white pairs  $(G,H)$  corresponding to such groups: we just have to list all the inequivalent fibred double covers of

$$(\mathbb{R}^3, V)/G = Q \xrightarrow{p} B .$$

We will see how to do this in §§5 and 6.

The remaining 50 groups are of two kinds. 36 of them fail to preserve any direction in  $\mathbb{R}^3$  at all. Thus, although they still correspond



to a 3-crystal

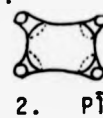
$$Q = \mathbb{R}^3/G,$$

$Q$  can no longer be given an  $S^1$ -fibred structure. Groups of this type (which constitute the cubic class mentioned in (2.5.1)) are dealt with in §6.

(3.6.1) This leaves 14 groups which give us the opposite problem: They preserve several directions in  $\mathbb{R}^3$ , none of which gives a unique fibration. The first two,

$$G \cong P1 \quad \text{and} \quad G \cong P\bar{1},$$

are the two most rudimentary crystallographic space groups in that they have the least structure.  $P1$  is generated by translations along any three linearly independent vectors;  $P\bar{1}$  is the same thing with an involution through the origin added to it. Thus they both preserve every choice of direction in  $\mathbb{R}^3$ , and consequently every rational foliation of  $\mathbb{R}^3$  by straight lines (i.e. passing through more than one lattice point) projects to the same  $S^1$ -fibration (see Table 4):



The first of these is just the orientable  $S^1$ -bundle over the torus and is a manifold. The double covers of these two trivial cases can be dealt with separately, and are done so in §7.

(3.6.2) The remaining 12 groups  $G$  are all members of the orthorhombic class ; they can be found by inspection of Table 4. Each preserves exactly three mutually perpendicular directions in  $\mathbb{R}^3$ , yielding the same fibred crystal three times over. Suppose that we have such a group  $G$  preserving  $V_\alpha, V_\beta, V_\gamma$  (all distinct) such that

$$(\mathbb{R}^3, V_\epsilon)/G \cong Q \xrightarrow{P} B$$

for  $\epsilon = \alpha, \beta, \gamma$ . Let  $(G, H_i)$  be a black and white group; then

$$(\mathbb{R}^3, V_\alpha)/H_i = Q_i \xrightarrow{P_i} B_i$$

is a fibred double cover of  $Q \xrightarrow{P} B$  and every fibred double cover corresponds to a black and white pair. However, unlike before, it is now possible for two such covers

$$(\mathbb{R}^3, V_\alpha)/H_i = Q_i \xrightarrow{P_i} B_i$$

and

$$(\mathbb{R}^3, V_\alpha)/H_j = Q_j \xrightarrow{P_j} B_j \quad (i \neq j)$$

to be inequivalent by Definition (3.3.1) but to correspond to equivalent black and white pairs. (Note that we *must* then have  $Q_j$  isomorphic to  $Q_i$ , though not necessarily fibred-isomorphic.) By choosing a different preserved direction  $V_\delta$  (so  $\delta = \beta$  or  $\gamma$ ), and looking at

$$(\mathbb{R}^3, V_\delta)/H_j = Q_j \xrightarrow{P_\delta} B_\delta,$$

it is possible for this to be equivalent, as a fibred double cover of

$Q \xrightarrow{p} B$ , to  $Q_i \xrightarrow{p_i} B_i$ ; consequently  $(G, H_i)$  and  $(G, H_j)$  would then be revealed as equivalent black and white groups, with the affine homeomorphism  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  of Definition (3.0) sending  $V_\alpha$  to  $V_\delta$ .

(3.6.3) How do we detect this behaviour in practice? First note that, because  $H_i \leq G$ , every  $H_i$  preserves the same directions  $V_\alpha, V_\beta$  and  $V_\gamma$  in  $\mathbb{R}^3$  as  $G$  (and perhaps more than just these). Secondly, because these three are mutually perpendicular, every  $Q_i = \mathbb{R}^3/H_i$  obtained as a double cover of  $Q$  will occur somewhere on the list as being fibred in the direction of the principal axis of  $H_i$  (see 2.5.1), and in at least one other direction if it gives an inequivalent covering (though, when  $H_i$  turns out to be monoclinic rather than orthorhombic, not necessarily in a third inequivalent direction even if it exists). Thus, given the list of inequivalent fibred double covers  $Q_i \xrightarrow{p_i} B_i$  of  $Q \xrightarrow{p} B$  (we will see how to produce this list in §§4 and 5), for every  $Q_i \xrightarrow{p_i} B_i$  which represents a non-principal choice of direction, we can pair it off with the  $Q_j \xrightarrow{p_j} B_j$  corresponding to the principal axis, which we know must also occur. Doing this, we can decide which distinct covers correspond to equivalent black and white groups and reduce the list accordingly. We will see an example of this in §7.

As we have just noted, the next task is: given an  $S^1$ -fibred 3-crystal, determine all of its fibred double covers up to equivalence. Looking back to (3.2.1), we observed there that these fall into two distinct classes which will be considered separately.

(3.7) Definition: Let  $\phi: Q' \rightarrow Q$  be a fibred double covering map of  $S^1$ -fibred 3-orbifolds, leading to a commutative diagram:

$$\begin{array}{ccc} Q' & \xrightarrow{\phi} & Q \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{\bar{\phi}} & B \end{array}$$

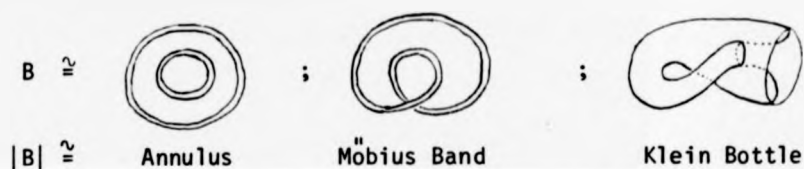
If  $\bar{\phi}: B' \rightarrow B$  is the projection of  $\phi$  to the base spaces then it is either a double covering of 2-orbifolds (and so  $B \cong B'/\bar{\rho}$ , where  $\bar{\rho}$  is the projection of the covering translation) or  $\bar{\phi}: B' \rightarrow B$  is an isomorphism (i.e.  $B' \cong B$ ). In the former case we call  $\phi: Q' \rightarrow Q$  a *D-cover* and in the latter case an *S-cover*.

//

The next two chapters deal with these in turn: D-covers in §4 and S-covers in §5.

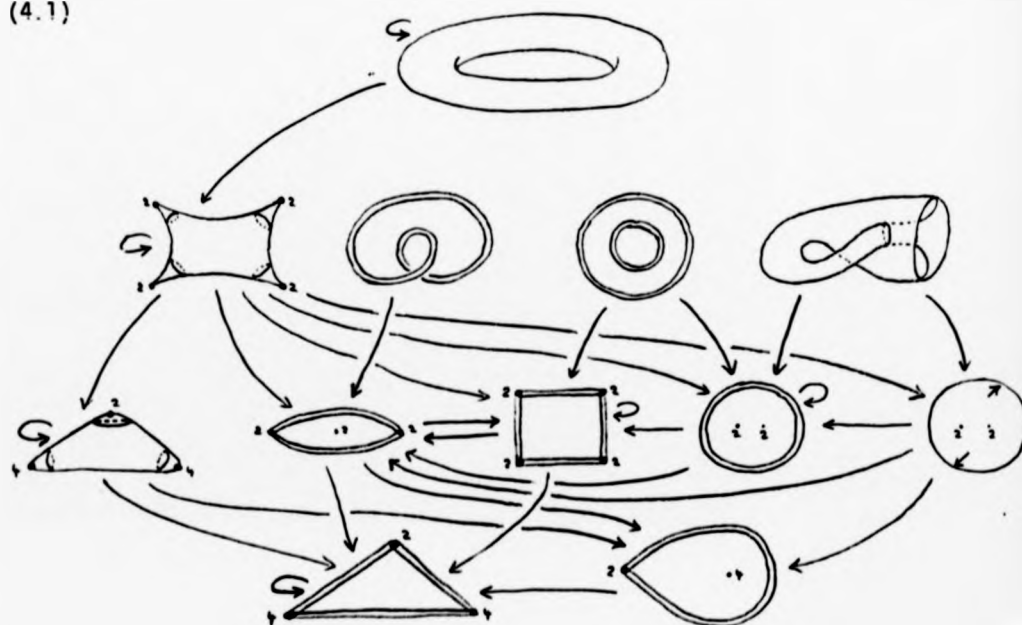
#### §4. Determination of D-covers.

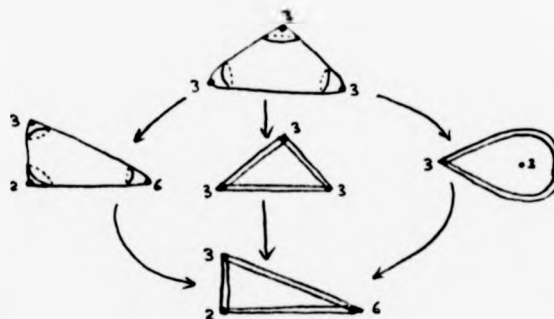
(4.0) We will informally refer to a D-cover as one formed by "unwrapping the base". Because certain groups  $G$  preserve more than one direction in  $\mathbb{R}^3$ , we need only consider 14 of the 17 2-crystals  $B$  as base (see Table 2). The excluded cases are:



Thus in finding D-covers of an  $S^1$ -fibred 3-crystal  $Q \rightarrow B$  we need only be able to construct double covers  $Q' \rightarrow B'$  for which  $\bar{\phi}: B' \rightarrow B$  is a double cover taken from one of the following two sub-diagrams of Table 1.

(4.1)





We proceed as follows: starting with  $Q \xrightarrow{p} B$ , for a given  $\bar{\phi}: B' \rightarrow B$  we wish to construct a commutative diagram:

$$\begin{array}{ccc} Q' & \xrightarrow{\phi} & Q \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{\bar{\phi}} & B \end{array}$$

where  $\phi: Q' \rightarrow Q$  is a fibred double covering map of  $S^1$ -fibred 3-crystals, projecting to  $\bar{\phi}: B' \rightarrow B$ .

First assume that the generic fibre of  $Q$  (and hence of  $Q'$ ) is  $S^1$ ; the case when it is  $mI$  is dealt with in (4.9). Then we have the following diagram of two short exact sequences (where  $\pi_1(Q) = G_Q$  etc.):

$$\begin{array}{ccccc} & & G_{B'} & & \\ & & \bar{\phi}_* \downarrow & & \\ \mathbb{Z} & \longrightarrow & G_Q & \xrightarrow{p_*} & G_B \\ & & & & \eta \downarrow \\ & & & & \mathbb{Z}_2 \end{array}$$

where  $\eta: G_B \rightarrow G_B/G_{B'} \cong \mathbb{Z}_2$  is the natural homomorphism.

Define:

$$G_{Q'} = p_*^{-1} \circ \bar{\phi}_* (G_{B'})$$

$$\phi_* = \text{id}|_{G_{Q'}} : G_{Q'} \rightarrow G_Q \quad (\text{so } \phi_* \text{ is injective})$$

$$p'_* = \bar{\phi}_*^{-1} \circ p_*|_{G_{Q'}} : G_{Q'} \rightarrow G_{B'} \quad (\text{so } p'_* \text{ is surjective})$$

We then have the following diagram of (horizontal and vertical) short exact sequences:

$$(4.2) \quad \begin{array}{ccccc} \mathbb{Z} & \longrightarrow & G_{Q'} & \xrightarrow{p'_*} & G_{B'} \\ \downarrow \cong & & \downarrow \phi_* & & \downarrow \bar{\phi}_* \\ \mathbb{Z} & \longrightarrow & G_Q & \xrightarrow{p_*} & G_B \\ & & \downarrow & & \downarrow \eta \\ & & G_Q/G_{Q'} & \xrightarrow[\cong]{\tau} & \mathbb{Z}_2 \end{array}$$

where  $\tau: G_Q/G_{Q'} \rightarrow \mathbb{Z}_2$  is defined by  $\tau(\alpha \cdot G_{Q'}) = \text{nop}_*(\alpha)$ .

(4.2.1) Lemma:  $\tau: G_Q/G_{Q'} \rightarrow \mathbb{Z}_2$  is a well-defined isomorphism.

Proof:  $\tau$  is well-defined since:

$$\begin{aligned} (\alpha \cdot G_{Q'} = \beta \cdot G_{Q'}) &\iff (\alpha\beta^{-1} \in G_{Q'}) \iff (p_*(\alpha\beta^{-1}) \in \bar{\phi}_*(G_{B'})) \dots \\ \dots &\iff (p_*(\alpha\beta^{-1}) \in \ker(\eta)) \iff (\text{nop}_*(\alpha\beta^{-1}) = \text{id}) \iff (\text{nop}_*(\alpha) = \text{nop}_*(\beta)) \end{aligned}$$

That  $\tau$  is an isomorphism comes from:

If  $\alpha \notin \phi_*(G_{Q'})$  (so  $\alpha \cdot G_{Q'} \neq \text{id} \in G_Q/G_{Q'}$ ) then  $p_*(\alpha) \notin \bar{\phi}_*(G_{B'})$ ,

so  $p_*(\alpha) \notin \ker(\eta)$  and therefore  $\tau(\alpha \cdot G_{Q'}) = \text{nop}_*(\alpha) \neq \text{id}$ .

(This shows that  $\tau$  is injective and surjective, since we are mapping into  $\mathbb{Z}_2$ .)

□

(4.2.2) Thus we see that a given  $\bar{\phi}:B' \rightarrow B$ , and hence a given  $\bar{\phi}_*:G_{B'} \rightarrow G_B$ , determines a unique  $\phi_*:G_{Q'} \rightarrow G_Q$  and hence a correspondingly unique  $S^1$ -fibred 3-orbifold:

$$(R^3, V)/G_{Q'} = Q' \xrightarrow{p'} B',$$

which is a fibred double cover of  $(R^3, V)/G_Q = Q \xrightarrow{p} B$ .

Given  $Q \rightarrow B$  and a double covering map  $\bar{\phi}:B' \rightarrow B$ , we start the process of constructing  $Q' \rightarrow B'$  by first determining its local structure using Table 3.1. Let  $U \cong D^2/\Gamma$  be a (small) co-ordinate chart in  $B$  such that  $\bar{\phi}^{-1}(U) \subset B'$  consists of components of the form  $D^2/\Gamma'$  (where  $\Gamma' \leq \Gamma$ ) and

$$p^{-1}(U) \cong (D^2 \times S^1)/\Gamma.$$

We can do this because, for small enough  $U$ ,  $p^{-1}(U)$  corresponds to one of the local fibre types 0 - 11 (see §1) and has one of the following two forms:

$$\begin{array}{ccc} (D^2 \times S^1)/\mathbb{Z}_\alpha & \text{or} & (D^2 \times S^1)/\Delta_{2\alpha} \\ \downarrow & & \downarrow \\ D^2/\mathbb{Z}_\alpha & & D^2/\Delta_{2\alpha} \end{array},$$

where both groups act fibre-preservingly on  $D^2 \times S^1$  and the projections to  $D^2$  are faithful.

Since  $\bar{\phi}:B' \rightarrow B$  is a double covering,  $\bar{\phi}^{-1}(U)$  consists of either a single connected component which double covers  $U$  (so  $|\Gamma:\Gamma'| = 2$ ) or



two disjoint copies of  $U$  (so  $\Gamma' = \Gamma$ ). The two cases correspond respectively to whether the chart is located at or away from the fixed-point set of the involution  $\bar{p}: B' \rightarrow B'$  which induces  $\bar{\phi}$ . In either case we have a well-defined  $W$ , unique up to fibred isomorphism, making

$$\begin{array}{ccc} W & \xrightarrow{\phi} & p^{-1}(U) \\ p'^{\downarrow} & & \downarrow \\ \bar{\phi}^{-1}(U) & \xrightarrow{\bar{\phi}} & U \end{array}$$

commute, where  $\phi$  is a fibred double covering map. When  $\Gamma' = \Gamma$ ,  $W$  is isomorphic to two (disjoint) copies of

$$p^{-1}(U) = (D^2 \times S^1)/\Gamma$$

and when  $|\Gamma:\Gamma'| = 2$  we have

$$W \cong (D^2 \times S^1)/\Gamma',$$

which we obtain from Table 3.1. Doing this for every chart on  $B$  we obtain the local structure of  $Q' \rightarrow B'$ . Note that the construction is unique for a given  $\bar{\phi}: B' \rightarrow B$  and that everything automatically agrees on the overlaps. If  $A' \subset B'$  is the fixed-point set for the involution on  $B'$  which generates  $\bar{\phi}: B' \rightarrow B$ , and  $A = \bar{\phi}(A') \subset B$  (we call  $A$  the *axis* for  $\bar{\phi}$ ), then we are producing  $Q' \rightarrow B'$  by *unwrapping*  $B$  along its axis  $A$  and doing the same for  $Q$  along  $p^{-1}(A)$ ; we shall resort to this loose terminology from time to time. The process is best understood by consulting the examples in §7.

The equivalence of double covers obtained in this way is the obvious one:

(4.3) Proposition: Given an  $S^1$ -fibred 3-orbifold  $Q \rightarrow B$ , two D-covers

$$\begin{array}{ccc} Q_1' & \xrightarrow{\phi_1} & Q \\ p_1' \downarrow & & \downarrow p \\ B_1' & \xrightarrow{\bar{\phi}_1} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} Q_2' & \xrightarrow{\phi_2} & Q \\ p_2' \downarrow & & \downarrow p \\ B_2' & \xrightarrow{\bar{\phi}_2} & B \end{array}$$

are equivalent (in the sense of Definition 3.2) exactly when there is a fibred isomorphism  $\psi: Q \rightarrow Q$ , orientation-preserving when  $Q$  is oriented, such that

$$p \circ \psi \circ p^{-1}(A_1) = A_2$$

where  $A_1 \subset B$  and  $A_2 \subset B$  are the axes of  $\bar{\phi}_1$  and  $\bar{\phi}_2$  respectively.

Proof: In one direction this is obvious, since if (3.2) holds then the axes are automatically sent to one another.

Going the other way, if we have a fibred  $\psi: Q \rightarrow Q$  sending one axis  $A_1$  to the other  $A_2$  in  $B$  then we can construct an isomorphism from  $Q$  unwrapped along  $p^{-1}(A_1)$  to  $Q$  unwrapped along  $p^{-1}(A_2)$  by first noting that, away from the axes, we simply have two copies of everything, and then "gluing in" over  $A_1$  and  $A_2$ .  $\square$

Note, in particular, that if two covers are to be equivalent then we must have  $B_1' \cong B_2'$ , though this is not in itself a sufficient condition -

appearances can be deceptive.

(4.4) Example: For the 3-crystal  $Q \rightarrow B$

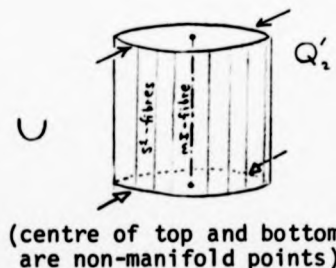
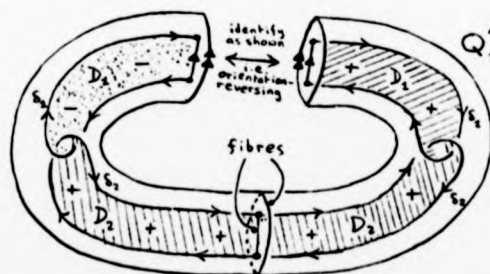
$$52 \text{ P. 111} : \quad \frac{1}{2} \text{ (diagram)} \rightarrow \frac{1}{2} \text{ (diagram)} \quad .$$

there is no fibred isomorphism which exchanges the two components of the Montesinos cycle (and consequently the covers obtained by unwrapping along each of the two sides of  $B$  are *not* equivalent).

Looking at how  $Q \rightarrow B$  is constructed, if we cut  $B$  along the broken line:



we get:



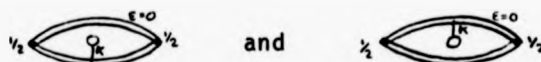
Both halves  $Q_1^1$  and  $Q_2^1$ , have as boundary a Klein bottle fibred by orientation-preserving circles.  $Q$  is formed by gluing the boundaries together, identifying fibres. The singular points which make up the Montesinos cycle (i.e. lie over  $\partial|B|$ ) sit as two linked circles in  $Q_1^1$ , labelled  $\delta_1$  and  $\delta_2$ . The mI-fibres attached to  $\delta_1$  and  $\delta_2$  form two intersecting discs,  $D_1$  and  $D_2$  respectively, which meet along the two mI-fibres lying over the dihedral points of  $B$ . Any fibred isomorphism of  $Q$  to itself which exchanges  $\delta_1$  and  $\delta_2$  must also exchange  $D_1$  and

$D_2$  ; but this is clearly not possible, for by orienting  $\delta_1$  and  $\delta_2$  and seeing how they cut  $D_2$  and  $D_1$  respectively we have:

$\delta_1$  cuts  $D_2$  from - to + and from - to +  
 whereas  $\delta_2$  cuts  $D_1$  from + to - and from - to + ,

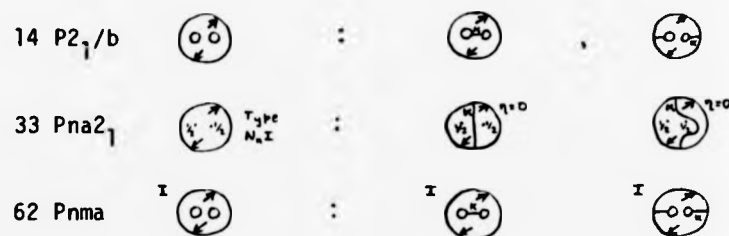
where we (arbitrarily) mark one side of each disc as '+' and the other as '-' . //

(4.4.1) This behaviour occurs as a result of the distinction between the classification of  $S^1$ -fibred 3-orbifolds with fixed base and the base-free classification. By and large these amount to the same thing, but not always: the above 3-crystal provides an example. With the fixed base classification [B + S:Y.5] it corresponds to the crystals:



which are *not* fibred-isomorphic by a map sending one base to the other *without rotating or reflecting it*. ( $K$  represents a suborbifold which, if we cut along it, leaves an orientable remainder.) Of course, without the restriction on the isomorphism, the one can be sent to the other by rotating through  $180^\circ$ .

There are very few other crystals which cause problems in this way. Besides some which have base a torus, for which the way that we construct their double covers takes such differences into account, there are only three. Together with their fixed base versions they are:



These crop up, respectively, in (4.7.6), (4.8.9) and (4.11.5). //

Returning to the construction of  $Q' \rightarrow B'$  as a double cover of  $Q \rightarrow B$ , so far we have determined  $Q'$  *only* locally; there are other invariants, relating to the non-local structure, which must also be calculated. Since we are looking at euclidean orbifolds, we know that we must have zero Euler number  $e_0(p')$ ; we also need to find:

- (4.5) Which boundary components of  $|B'|$  (if any) are marked by '\*' 's?
- (4.6) What parities are assigned to the Montesinos cycles (when these are not redundant or determined by other factors)?
- (4.7) When no '\*' 's are assigned in (4.5), there are no type 4 cone points, and  $|B'| \neq D^2, S^2, \text{Ann.}$ , which Seifert type is assigned to  $Q' \rightarrow B'$ ?
- (4.8) When (4.7) assigns a Seifert type other than  $\underline{0}$ , and  $\partial|B'|$  consists only of (orientation-preserving) Montesinos cycles or is empty, which  $n \in \mathbb{Z}_2$  do we assign to  $Q'$ ?

These are resolved as follows:

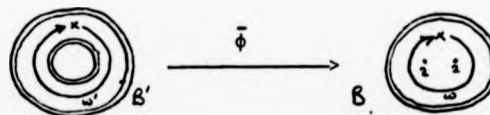
- (4.5)  $B'$  can be any one of the 17 2-crystals. Of these, 16 have underlying space  $|B'|$  with either empty boundary or a single boundary component;

consequently (see 2.3(ii)), the presence of  $'**'$  s is completely determined by the number of type 4 fibrations in the type diagram for  $Q' \rightarrow B'$  i.e. by the local structure (4.2.2). Only in the remaining case, when  $|B'| = \text{annulus}$ , do we have any work to do; note that either both boundary components of  $|B'|$  are marked by  $'**'$  s or neither is.

When  $|B'|$  is an annulus the presence of  $'**'$  s on  $\partial|B'|$  indicates that  $p'^{-1}(|B'| - \partial|B'|)$  is non-orientable (as a manifold). This is equivalent to saying that a closed path  $\omega': S^1 \rightarrow B' - \partial|B'|$  has a Klein bottle  $p'^{-1}(\omega') \subset Q'$  as pre-image, as opposed to a torus for the orientable case. We shall refer to such a path as orientation-reversing, meaning that it lifts to an orientation-reversing path in  $Q'$ .

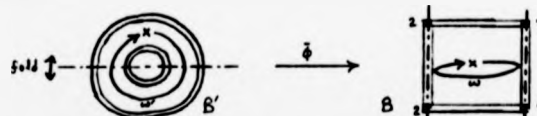
A glance at (4.1) shows that there are only two coverings  $\bar{\phi}: B' \rightarrow B$  which we need to consider:

(4.5.1)



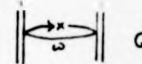
A path  $\omega$  going once around  $\partial|B|$  lifts to a suitable  $\omega'$  going once around  $B'$ . Thus  $\omega'$  is non-orientable exactly when  $\omega$  is, i.e. exactly when there is a  $'**'$  on  $\partial|B|$  (or equivalently when  $Q \rightarrow B$  has exactly one type 4 fibration).

(4.5.2)



There are two sides of  $B$  about which we unfold. A path  $\omega$  lifting to an appropriate  $\omega'$  is one which meets each of these sides exactly once. There are three distinct cases:

(i) Both sides consist of type 2 fibrations (in  $Q$ ) :



Then  $p^{-1}(\omega)$  is a *Corway sphere*:



This cannot be covered by a Klein bottle and so must lift to a torus in  $Q'$ , which therefore possesses no  $'*'$  s in its type diagram.

(ii) One side consists of type 2 fibrations, the other of type 5 or 6 :

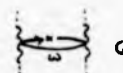


Then  $p^{-1}(\omega)$  looks like:



where the  $mI$ -fibre lies over the type 2 points and the outer  $S^1$ -fibre is a mirror (or a mirror with  $\pi$ -rotation) lying over the type 5 (or type 6) points.  $p'^{-1}(\omega')$  is formed by doubling along the outer fibre (together with a  $\pi$ -rotation in the type 6 case) and unfolding by a rotation about the  $mI$ -fibre. Thus we have an orientation-reversing action and an orientation-preserving one; we therefore get a Klein bottle and so  $a|B'|$  has  $'*'$  s.

(iii) Both sides are of type 5 or type 6 :



This time we have two orientation-reversing actions whose combined effect is to make  $p'^{-1}(\omega)$  a torus, hence there are no  $'*'$  s on  $a|B'|$ .

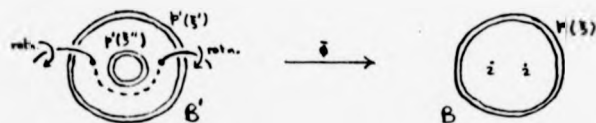
(4.6) Now we need to find the parities of any Montesinos cycles in  $Q' \rightarrow B'$ .

(4.6.1) First suppose that  $\xi' \subset Q'$  is a Montesinos cycle which is lifted as a copy of such a cycle  $\xi$  in  $Q$  i.e. there is another cycle  $\xi'' \subset Q'$  with  $\xi' \cap \xi'' = \emptyset$  such that

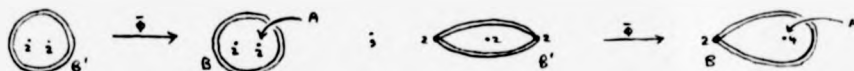
$$\xi' \cup \xi'' = \phi^{-1}(\xi)$$

Then the parity  $\epsilon(\xi')$  (and of course  $\epsilon(\xi'')$ ) is clearly equal to  $\epsilon(\xi)$ .

This takes care of the case:



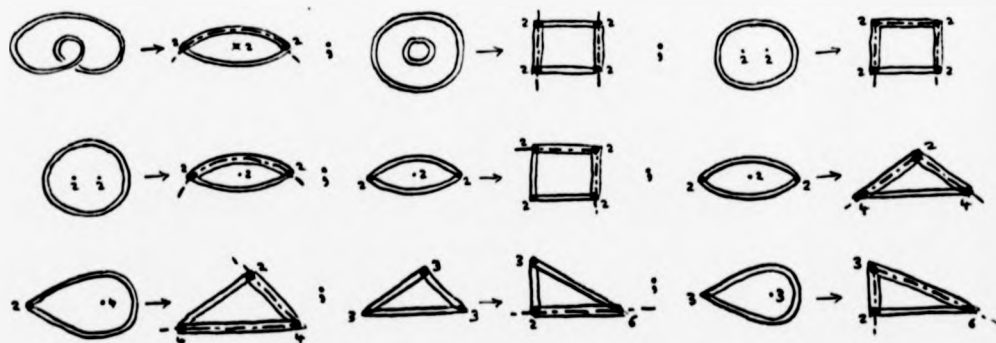
(4.6.2) Now assume that (4.6.1) does not happen but that  $\phi(\xi') = \xi$  is still an entire Montesinos cycle in  $Q$ . This affects the two covers:



where, for both, we are unwrapping about a single cone point (marked  $A$ ).

We immediately see that  $\xi \cap p^{-1}(A) = \emptyset$  and that the covering involution  $\bar{p}: B' \rightarrow B'$  which generates  $\bar{\phi}$  acts as a rotation on  $p'(\xi')$ . Hence the interval bundle over  $p'(\xi')$  by which we determine  $\epsilon(\xi')$  (see [B + S: Y.3]) must be an annulus i.e.  $\epsilon(\xi') = 0$ , since a Möbius band cannot double cover a Möbius band or an annulus without fixed points.

(4.6.3) The remaining cases are those for which  $\xi' = \phi^{-1}(\gamma)$ , where  $\gamma \subset Q$  is *not* the whole of a Montesinos cycle. This involves the covers:



(we unwrap about  $A \subset B$  marked by — — —'s and x's). Working in  $B$  we see that



$$\partial(p(\gamma)) = p(\gamma) \cap A,$$

with the endpoints of  $p(\gamma)$  being corners of  $B$ ; they are distinct from one another, else we are in the previous case (4.6.2).

Over these endpoints the following three local fibre types can occur in  $Q \rightarrow B$ :

Type 10 :



Type 11 :

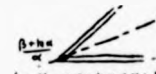
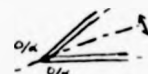


Type 3 :

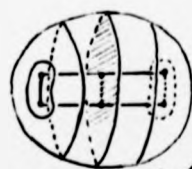


where in each case the horizontal portion is part of  $p(\gamma)$  and the corner is an endpoint of  $p(\gamma)$ . Note that the other part of  $\partial|B|$  incident on such an endpoint must be part of  $A$ , the image in  $B$  of the fixed-point set of  $\bar{p}:B' \rightarrow B'$ .

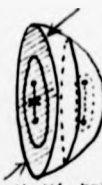
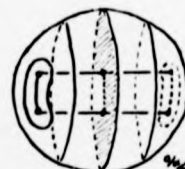
The double covers of these three objects which concern us here are:



The involution  $\rho:Q' \rightarrow Q'$  acts by reflection on the base  $B'$ . On the fibres, for the first object it acts as the identity on regular fibres (and hence also on the mI-fibre over the corner); for the second it acts by rotation of fibres through  $\pi$ , and hence restricts to reflection of the corner mI-fibre through its midpoint. The picture for these first two are:



reflect across vertical plane.



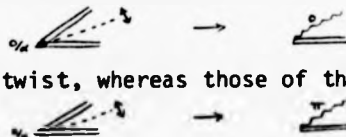
reflect in vertical plane and rotate each fibre in its end.

identify points on shaded face by antipodal map through 'u'.

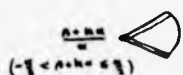
We construct  $\xi' \subset Q'$  by lifting  $\gamma$  from  $Q$  in two stages. First, the part of  $\gamma$  away from its ends lifts as two disjoint copies of itself

(up to *orientation-reversing* fibred isomorphism i.e. they may be mirror images). Thus any twists in  $\gamma$  negate one another in  $\xi'$  and, in particular, if  $\gamma \subset \xi$  (where  $\xi$  is a Montesinos cycle in  $Q$ ) then  $\epsilon(\xi)$  has no bearing on  $\epsilon(\xi')$ . Hence  $\epsilon(\xi')$  is determined entirely by the double covers over the two ends of  $\gamma$ .

The second stage, since we already know what the local double covers are, is to determine which of them introduce a twist to  $\xi'$  i.e. contribute 1 to  $\epsilon(\xi')$ . To understand this we must look at the local structure. It is easier to view matters from the point of view of the local fibre type in  $Q' \rightarrow B'$  covering something in  $Q \rightarrow B$  (by "folding up") than the other way round: we are looking for which parts of the singular set in  $Q'$  are identified by the covering involution. We immediately see that covers like

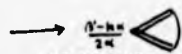


do not introduce a twist, whereas those of the form



(so  $k$  is exactly one of  $-1, 0, +1$ )

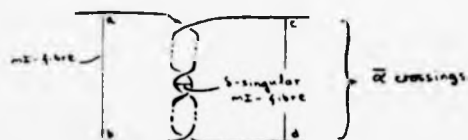
is the same, after putting  $\beta' = \beta + k\alpha$ , as



(so  $k=0$  and one of  $+1, -1$ )

We divide into the two cases of  $\bar{\alpha}$  even and  $\bar{\alpha}$  odd, just as in (1.1.3) (where  $\delta = (\alpha, \beta) = (\alpha, \beta + k\alpha) = (\alpha, \beta')$  and  $\bar{\alpha} = \alpha/\delta$ ). The fibration

looks like:



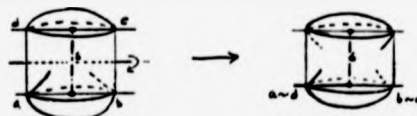
(i)  $\bar{\alpha} = \text{even}$ : Re-drawn with the singular set horizontal (and the fibres lying on the boundary Conway sphere having slope infinity) this looks like:



and folds about a horizontal axis as shown to form  $\frac{a'}{a''}$ . Note that the involution sends  $a$  to  $c$  (and  $b$  to  $d$ ). This is what we are interested in; the topological square over this corner, with boundary (on the Conway sphere) consisting of arcs of slope 0 or  $\infty$ , is exactly the square  $acdb$  (with vertices in that order), and so automatically glues, without a twist, to what we have already lifted.

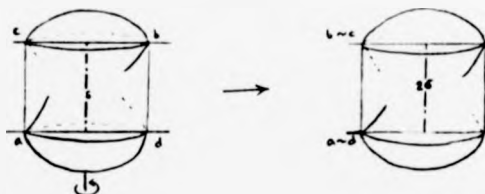
Now look at  $\frac{a'}{a''} \rightarrow \frac{a'-ka}{a''} \quad (k = \pm 1 \text{ or } -1 \text{ such that } -a \leq a' - ka \leq a)$

To obtain this, we must first twist the picture on the left and then fold:



so  $a$  is sent to  $d$  (and  $b$  to  $c$ ). Therefore, if we are to glue this to the already constructed part of  $\xi'$ , we must put in a twist.

(ii)  $\bar{\alpha} = \text{odd}$ : This case is similar: when  $k = 0$ , it is exactly the same; when  $k = 1$ , the involution looks like:



i.e. we rotate about a vertical axis, hence doubling the singularity of the  $m1$ -fibre over the corner. Again,  $a$  is sent to  $d$  (and  $b$  to  $c$ ), so we need a twist.

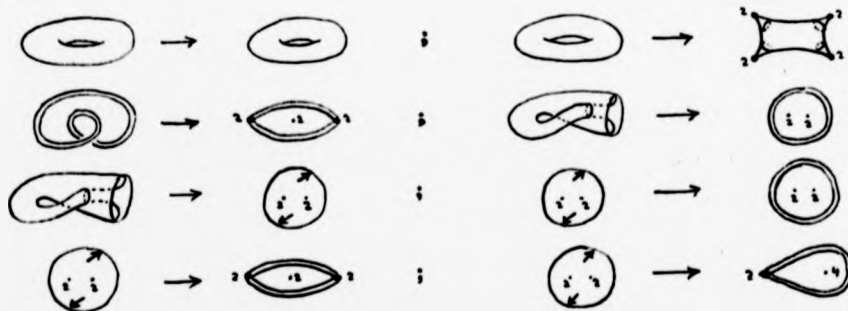
Combining the two cases, we have:

$$\epsilon(\xi') = \#(\text{local coverings of form } \triangleleft \rightarrow \triangleleft \text{ and } \triangleleft \rightarrow \triangleleft) \pmod{2}$$

In the euclidean case, the latter are exactly the following:

$$0/1 \rightarrow 1/2, \quad 0/2 \rightarrow 2/4, \quad 0/3 \rightarrow 3/6, \quad -1/3 \rightarrow 2/6, \quad 1/3 \rightarrow -2/6.$$

(4.7) Now we wish to determine the Seifert type of  $Q' \rightarrow B'$  when there are no type 4 fibrations in its local structure and when no  $***$ 's have been assigned in (4.5). The only closed euclidean  $S^1$ -fibred 3-orbifolds which can have Seifert type other than  $\underline{0}$  are those whose base has underlying space a torus, Möbius band, Klein bottle or projective plane. Therefore, after consulting (4.1), we are concerned with only the following eight covers  $\bar{\phi}: B' \rightarrow B$



The Seifert type of  $Q' \rightarrow B'$  represents a homology class

$[K'] \in H_1(|B'|, \partial|B'|; \mathbb{Z}_2)$  so that, if the type is not  $\underline{0}$ , then  $p'^{-1}(\text{int}|B'|)$  is

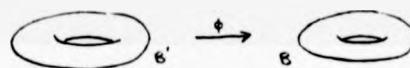
non-orientable as a manifold (and note that it *is* a manifold, since we are assuming the absence of type 4 fibrations), whereas the 3-manifold

$$p'^{-1}(\text{int } |B'| - K')$$

formed by cutting along  $p'^{-1}(K')$  *is* orientable.

The idea, then, is to determine how the orientability of  $p'^{-1}(\text{int } |B'|)$  is related to the nature of  $p'^{-1}(K')$  and, knowing how  $K'$  and  $K = \bar{\phi}(K')$  lie in  $B'$  and  $B$  respectively, to see how  $p^{-1}(K) \subset Q$  lifts into  $Q'$  in the light of this.

(4.7.1)



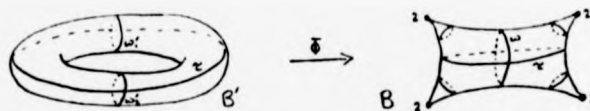
First, note that:

$Q'$  is of Seifert type No  $\iff$  There exists a simple closed curve  $\omega'$  in  $B'$  with  $p'^{-1}(\omega') = \text{Klein bottle}$ .

If  $Q \rightarrow B$  is of Seifert type 0 then so must  $Q' \rightarrow B'$  be, since Klein bottles cannot double cover tori so we cannot create any in  $Q'$  from the tori lying over all essential curves  $K \subset B$ .

If  $Q \rightarrow B$  is of Seifert type No then there is a simple closed curve  $\omega \subset B$  such that  $p^{-1}(\omega) \subset Q$  is a Klein bottle (fibred by orientation-preserving circles) and  $p^{-1}(B - \omega)$  is orientable. Depending on how  $B'$  covers  $B$ ,  $\omega$  either lifts to two disjoint copies of itself in  $Q'$ , whence  $Q'$  must be of Seifert type No, or  $\omega$  is double covered by a single simple closed curve  $\omega' \subset B'$ , for which  $p'^{-1}(\omega')$  must then be a torus. Since no new Klein bottles can have been created in  $Q'$ ,  $Q' \rightarrow B'$  must be of type 0.

(4.7.2)



where  $\bar{\phi}^{-1}(\omega) = \omega_1' \cup \omega_2'$  and  $\bar{\phi}^{-1}(\tau) = \tau'$ .

If the type diagram for  $Q \rightarrow B$  is:



i.e. there are no type 4 cone points, then  $Q$  is orientable and hence so is  $Q'$ , whence  $Q' \rightarrow B'$  has Seifert type 0.

If the type diagram for  $Q \rightarrow B$  is:



i.e. all the fibrations over the cone points are of type 4, then  $p^{-1}(\omega) = p^{-1}(\tau) = \text{torus}$  and hence

$$p'^{-1}(\omega_1') = p'^{-1}(\omega_2') = p'^{-1}(\tau') = \text{torus},$$

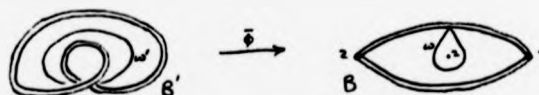
so  $Q'$  is orientable and  $Q' \rightarrow B'$  again has Seifert type 0.

If the type diagram for  $Q \rightarrow B$  is:



i.e. there are exactly two type 4 fibrations, then the Seifert type of  $Q' \rightarrow B'$  is  $N_0$ : if  $p^{-1}(\omega)$  is a Klein bottle (i.e.  $\omega$  cuts  $B$  into two halves, each containing a point of either fibre type) then  $p'^{-1}(\omega_1')$  is a Klein bottle, so  $Q' \rightarrow B'$  has type  $N_0$ ; if  $p^{-1}(\omega)$  is a torus then  $p^{-1}(\tau)$  is a Klein bottle, as is  $p'^{-1}(\tau')$ , and so  $Q' \rightarrow B'$  has type  $N_0$ .

(4.7.3)



$\bar{\phi}^{-1}(\omega) = \omega'$ , where  $\omega$  is a closed path going once around  $B'$  and  $\omega$  meets the mirror along which we unfold  $B$  in exactly one point.

Clearly  $p'^{-1}(\text{int}|B'|)$  is orientable exactly when  $p'^{-1}(\omega')$  is a

Klein bottle, so we need to know when  $p^{-1}(\omega) \subset Q$  lifts to a Klein bottle in  $Q'$  (making  $Q' \rightarrow B'$  of Seifert type  $Q$ ) and when it lifts to a torus (making  $Q' \rightarrow B'$  of Seifert type  $N_n I$ ). To see this we consider

$$\omega = \omega_1 \cup \omega_2 \quad \text{in:} \quad \begin{array}{c} \text{---} \omega_2 \text{---} \\ \text{---} \omega_1 \text{---} \end{array} \quad ,$$



and look at  $p^{-1}(\omega_1)$  and  $p^{-1}(\omega_2)$ .

$p^{-1}(\omega_1)$  is a torus or a Klein bottle (fibred by orientation-preserving circles) respectively when the fibration in  $Q$  over the cone point is of type 1 or type 4.

$p^{-1}(\omega_2)$  is one of three types:

Type 2:  ; Type 5:  ; Type 6:    
  $\cong$  Möbius band

(4.7.4) Proposition: These lift into  $Q' \rightarrow B'$  subject to the following table:

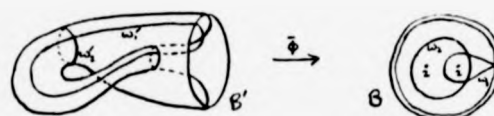
	Cone point is type 1	Cone point is type 4
$\omega_2$ meets  (type 2)	$Q' \rightarrow B'$ has Seifert type $Q$	$Q' \rightarrow B'$ has Seifert type $N_n I$
$\omega_2$ meets  (type 5 or 6)	$Q' \rightarrow B'$ has Seifert type $N_n I$	$Q' \rightarrow B'$ has Seifert type $Q$

Proof: If  $\omega_2$  meets a type 2 fibration over  $a|B|$  then the covering is:

$$\begin{array}{c} \text{---} \omega_2 \text{---} \\ \text{---} \omega_1 \text{---} \end{array} \xrightarrow{\phi} \begin{array}{c} \text{---} \omega_2 \text{---} \\ \text{---} \omega_1 \text{---} \end{array} = \begin{array}{c} \text{---} \omega_2 \text{---} \\ \text{---} \omega_1 \text{---} \end{array} p^{-1}(\omega_1)$$

So, if  $p^{-1}(\omega_1) = \text{torus}$ , cutting it and gluing in the double cover of  $p^{-1}(\omega_2)$  in a consistent fashion gives  $p^{-1}(\omega') = \text{Klein bottle}$ . Altering either condition (on  $\omega_1$  or  $\omega_2$ ) changes the gluing and makes  $p^{-1}(\omega')$  a torus; altering both simultaneously changes this back to a Klein bottle.  $\square$

(4.7.5)



First notice that by using the definitions of  $\underline{0}$ ,  $N_n I$  and  $N_n II$ , the following table relates the Seifert type of  $Q' \xrightarrow{p'} B'$  to the nature of  $p'^{-1}(\omega_1')$  and  $p'^{-1}(\omega_2')$ :

	$p'^{-1}(\omega_1') = \text{torus}$	$p'^{-1}(\omega_2') = \text{Klein bottle}$
$p'^{-1}(\omega_1') = \text{torus}$	Seifert type $N_n I$	Seifert type $N_n II$
$p'^{-1}(\omega_2') = \text{Klein bottle}$	Seifert type $\underline{0}$	

From this, and observing that  $\omega_2 \subset B$  lifts to two copies of itself in  $B'$  (and hence anything which fibres over  $\omega_2$  lifts to two copies in  $Q'$ ), we can immediately see that

$Q' \rightarrow B'$  is of Seifert type  $N_n II \iff p'^{-1}(\omega_2') = \text{Klein bottle} \iff Q \rightarrow B$  has exactly one type 4 fibration  $\iff$  There is a  $\ast\ast$  on  $\partial|B|$ .

Now suppose that  $p^{-1}(\omega_2)$  is a torus (and hence so is  $p'^{-1}(\omega_2')$ ). Then, since the fibres in  $Q \rightarrow B$  over the two cone points are of the same type i.e. both type 4 or both type 1, we immediately see that there is no ambiguity about which cone point is enclosed by  $\omega_1$ . Using the same argument as for (4.7.4), we conclude that:

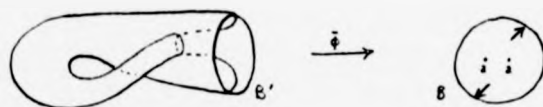
$p'^{-1}(\omega_1')$  is a torus exactly when  $\begin{cases} \text{the cone points of } Q \rightarrow B \text{ are type 1 and } \partial|B| \text{ comes from } \cup \\ \text{the cone points of } Q \rightarrow B \text{ are type 4 and } \partial|B| \text{ comes from } \cap \end{cases}$

To summarize:

If the weak type diagram of $Q \rightarrow B$ is:	then $Q' \rightarrow B'$ is of Seifert type:
or	$N_n II$
or	$N_n I$
or	$\underline{0}$



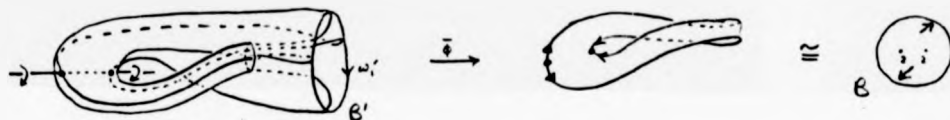
(4.7.6)



The first thing to notice is that  $Q' \rightarrow B'$  cannot be of Seifert Type  $N_n II$ ; for if it were then we should have  $p'^{-1}(\omega_1') = \text{Klein bottle}$ , where  $\omega_1'$  lies in  $B'$  as:



- see the table at the start of (4.7.5). But the cover  $\bar{\phi}: B' \rightarrow B$  operates by rotating the "top" Möbius band of the Klein bottle onto the "bottom" one, with two fixed points, viz:



Isotoping  $\omega_1'$  to the position shown, the action of the involution on it is just a rotation (through half its own length). Thus if  $\omega = \bar{\phi}(\omega_1') \in B$  then, regardless of whether  $p^{-1}(\omega_1) \subset Q$  is a torus or a Klein bottle, its pre-image in  $Q' \xrightarrow{p'} B'$ , fibred over  $\omega_1'$ , must be a torus  $\otimes$ .

Now that we have shown that  $Q' \rightarrow B'$  cannot be of Seifert type  $N_n II$  we must determine when it is of type  $\underline{0}$  and when of type  $N_n I$ . Let  $\omega_2'$  be a closed path going once around the Klein bottle  $B'$ , viz:



Then  $\omega_2 = \bar{\phi}(\omega_2') \in |B|$  is a non-trivial closed path in the projective plane  $|B|$ . If we assume that both the local fibre types over the cone points of  $B$  are type 1, i.e. orientable, then, regardless of how  $\omega_2$  lies in  $B$  with respect to them, we have:

$$p^{-1}(\omega_2) = \begin{cases} \text{torus} & \text{when } Q \rightarrow B \text{ is of Seifert type } N_n I \\ \text{Klein bottle} & \text{when } Q \rightarrow B \text{ is of Seifert type } \underline{0} . \end{cases}$$

But in choosing a suitable  $\omega_2 \subset B$  we can ensure that it lifts to two disjoint copies of itself in  $B'$ , one of which is  $\omega_2'$ . Hence the fibred 2-orbifold lying over  $\omega_2$  does the same and we have, when the fibres over the cone points of  $Q \rightarrow B$  are both of type 1, that

$$Q' \rightarrow B' \text{ has Seifert type } \left\{ \begin{smallmatrix} 0 \\ N_n I \end{smallmatrix} \right\} \text{ exactly when } Q \rightarrow B \text{ has Seifert type } \left\{ \begin{smallmatrix} 0 \\ N_n I \end{smallmatrix} \right\}$$

Since the case of two different fibre types cannot occur, the only remaining case to consider is when  $Q \rightarrow B$  has weak type diagram:



Now the Seifert type of  $Q' \rightarrow B'$  depends on the nature of the covering  $\bar{\phi}: B' \rightarrow B$  (see 4.4.1). This is best seen by looking at two closed paths  $\tau_1$  and  $\tau_2$  lying in  $B$  as follows:



Because the boundary of a neighbourhood of a type 4 fibre is a Klein bottle, it is clear that if  $p^{-1}(\tau_1)$  is a torus then  $p^{-1}(\tau_2)$  must be a Klein bottle and vice versa. So when we have this situation, there are two distinct covers  $Q_1' \rightarrow B'$  and  $Q_2' \rightarrow B'$  corresponding to whether

$$\bar{\phi}(\omega_2') = \tau_1 \text{ or } \tau_2.$$

One cover is of Seifert type  $\underline{0}$ , the other of Seifert type  $N_n I$ . These are the only two inequivalent covers with base  $B' = \text{Klein bottle}$ , since given any other closed path  $v$  in  $B$  such that  $v$  is non-trivial in  $|B|$ , we can construct a fibred automorphism of  $Q$  whose projection to the base sends  $v$  to  $\tau_1$  or  $\tau_2$ .

(4.7.7)



We need only assign a Seifert type to  $Q' \rightarrow B'$  when it has two type 1 fibres and hence when  $Q \rightarrow B$  has at least one type 1 fibre (lifting to two copies of itself in  $Q' \rightarrow B'$ ). With  $\omega_1$  and  $\omega_2$  lying in  $B$  as shown, and using the same argument as (4.7.4), we get exactly the same table as occurs there (we can interpret the headings as "both fibres in  $Q \rightarrow B$  are of type 1" and "one fibre of  $Q \rightarrow B$  is type 1, the other type 4").

(4.7.8)



Again we have the same table. Note that there is no ambiguity about which side is met by  $\omega_2$ , since if they are distinct in the weak type diagram i.e. we have:

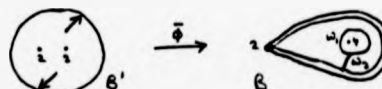


then these both lift to the weak type diagram:



for which no Seifert type is required.

(4.7.9)



Again we have the same table.

(4.8) We now need to determine the obstruction  $n \in \mathbb{Z}_2$  in the relevant cases. These are exactly the covers  $\bar{\phi}: B' \rightarrow B$  already listed in (4.7). Because the definition of  $n \in \mathbb{Z}_2$  depends on whether  $|B'|$  has empty boundary or not (see 2.1.7), we deal with these cases separately:

$$\begin{aligned} Q' \rightarrow B' \text{ has } \partial|B'| \neq \emptyset & \quad (\text{so } |B'| \cong \text{Möbius band}) \\ Q' \rightarrow B' \text{ has } \partial|B'| = \emptyset & \quad (\text{so } |B'| \cong \mathbb{RP}^2, T^2 \text{ or } K^2), \end{aligned}$$

(4.8.1)  $Q' \rightarrow B'$  has  $\partial|B'| \neq \emptyset$ , so  $\bar{\phi}: B' \rightarrow B$  looks like:



where  $C' = \partial|B'|$  lifts to an (orientation-preserving) Montesinos cycle in  $Q'$ . There is a standard 2-fold section  $t$  of  $p'$  over  $\partial N(C')$ , since  $N(C')$  looks like:



Choosing a 1-fold section  $s$  of  $p'$  over  $\partial N(C')$ , we define

$$\epsilon'(C') = s \cdot t \in \mathbb{Z}$$

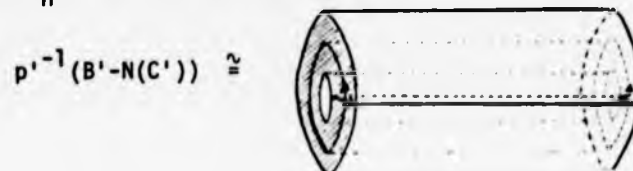
i.e. the homological intersection number of  $s$  and  $t$  in  $\partial N(C')$ .

Since we have

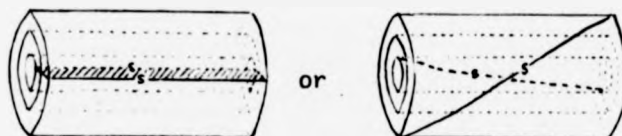
$$\epsilon(C') = \begin{cases} 0 \\ 1 \end{cases} \text{ depending on whether } p'^{-1}(C') \cong \begin{cases} \text{annulus} \\ \text{Möbius band} \end{cases},$$

we can obviously choose the section  $s$  so that  $\epsilon'(C') = \epsilon(C')$ ; we will always assume that this is so. There then may or may not be an obstruction to extending  $s$  to a 1-fold section over all of  $B' - N(C')$ ; because  $Q'$  is non-orientable, this obstruction  $n$  is easily seen to lie in  $\mathbb{Z}_2$ .

For us to have to determine  $n$  at all,  $Q' \rightarrow B'$  must be of Seifert type  $N_n I$ , in which case we have



It is how  $p'^{-1}(N(C'))$  is glued to this which determines whether  $n = 0$  or  $n = 1$ . In the above picture,  $s$  can lie as either:



The first can be extended to a 1-fold section (the shaded Möbius band); the second cannot. These are the only two possibilities - anything else is fibred-isomorphic to one of them.

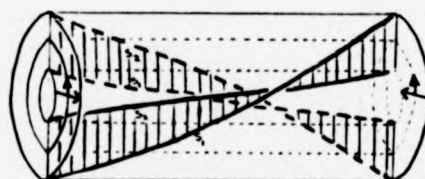
When  $\epsilon(C') = 1$ , i.e.  $p'^{-1}(C')$  is a Möbius band, the  $n = 0$  and  $n = 1$  cases are equivalent by 2.1.8(c) and  $(d_3)$ , so we need only determine  $n$  for the case when  $\epsilon(C') = 0$ . We wish to know whether a 1-fold section which has slope zero on  $\partial N(C')$  can be extended to a 1-fold section over all of  $B' - N(C')$ . If so then  $n = 0$ ; if not then  $n = 1$ . Another way of distinguishing the two cases is as follows:

(4.8.2) Proposition: Let  $p:M \rightarrow X$  be a non-orientable (manifold)  $S^1$ -bundle over a Möbius band  $X$  and let  $s$  be a 1-fold section of the fibred torus  $p^{-1}(\partial \bar{X})$ . Then:

- either (i)  $s$  can be extended to a global 1-section;  
or (ii) there exists a connected 2-fold section of the bundle  
 (i.e. an annulus) whose boundary is two copies of  $s$ ,

the two cases being mutually exclusive.

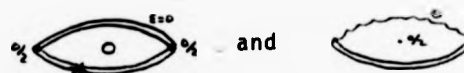
Proof: As we have already noted, for a given section  $s$  of  $p^{-1}(\partial \bar{X})$  the obstruction to extending it to a global section lies in  $\mathbb{Z}_2$ . If this obstruction is zero we have case (i); if  $n \in \mathbb{Z}_2$  is equal to 1 then there does not exist a global 1-fold section whose boundary is  $s$ . We have:



where  $s_1$  and  $s_2$  are two copies of  $s$ . Clearly neither on its own extends to a global section, but together they form the boundary of the shaded annulus which sits 2-fold over the base. It remains to show that if  $s$  extends to a 1-section then such a 2-section cannot exist; this is made obvious by drawing the picture for  $n = 0$ .  $\square$

So in order to decide whether  $Q' \rightarrow B'$  has  $n = 0$  or  $n = 1$  we need only look for the existence of sections with (one or more copies of) a given boundary. This is conveniently done by examining the lifts of (partial) sections of  $Q \rightarrow B$ . The following examples cover all the crystals  $Q \rightarrow B$  for which  $|B|$  lifts to a Möbius band and for which we need to determine  $n \in \mathbb{Z}_2$ :

(4.8.3) Examples:



are both covered by:

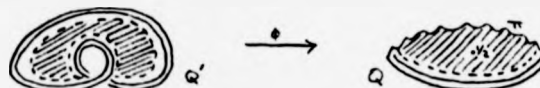


(as opposed to the same thing but with  $n = 1$ ). This is because both the fibred 3-crystals  $Q \rightarrow B$  admit a partial 1-fold section which lifts to a partial section of  $Q' \rightarrow B'$  in such a way that it can be completed to 1-fold section over  $B' - N(C')$  (where  $C' = \partial|B'|$ ) whose boundary in  $p'^{-1}\partial N(C')$  has null intersection with the standard 2-fold section inherited from  $p'^{-1}(C')$ : i.e.  $\epsilon'(C') = 0$  and  $n = 0$ .

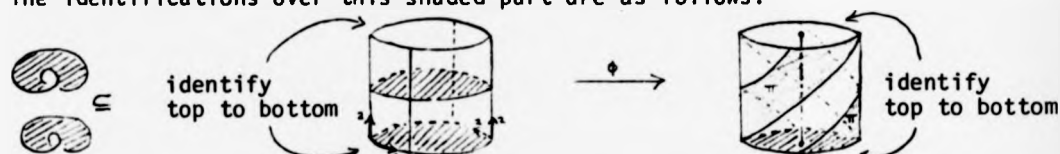
If



then we are interested in what happens over the shaded section, which lifts to the part of the Möbius band away from its boundary, viz:



The identifications over this shaded part are as follows:



i.e. the 2-fold section lifts to *two* copies of a 1-fold section. It can be easily checked that this can be done in such a way that the boundary of the section does not intersect the standard one i.e. we have  $n = 0$  again.

Using a similar argument to the above, the following 3-crystals all admit partial 2-fold sections which lift to a connected 2-fold section in  $Q' \rightarrow B'$ :



and they are therefore covered by



This leaves only the following:



In (4.4) we noted that there is no fibred automorphism of this crystal

which exchanges the two components of the Montesinos cycle. So it is clear that the covers obtained by unwrapping along each of these two components respectively are *not* equivalent (in the sense of Definition (3.3.1)), since such an equivalence would yield the required fibred automorphism. Although it does not in general follow that the two possibilities for  $Q' \rightarrow B'$  obtained in this way are different (they may only be different *as covers of*  $Q \rightarrow B$ ), in this case they happen to be distinct. Looking more closely at  $Q \rightarrow B$ , we have:



Unfolding the  $D_1$ -component of the Montesinos cycle yields a 1-section of  $Q' \rightarrow B'$  whose boundary  $s \subset p'^{-1}\partial N(C')$  lifts from one of the two components of  $t$  and is therefore "horizontal" i.e.  $n = 0$ . Re-drawing this picture with  $D_2$  at centre gives:



the significant difference being in how the components of the singular set cross one another. In turn, when we now unfold along the  $D_2$ -component, a "horizontal" 1-section must have boundary which cuts the lift of  $t$  and so we get  $n = 1$ .

(4.8.4) Now we must deal with the cases when  $Q' \rightarrow B'$  has  $\partial|B'| = \emptyset$  i.e.



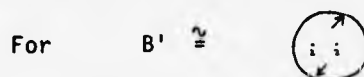


$\eta$  is defined only when  $Q' \rightarrow B'$  is non-orientable with no type 4 cone fibres; so in particular  $Q' \rightarrow B'$  does not have Seifert type 0.

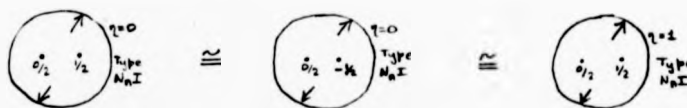
$\eta = 0$  if there exists a 1-fold section over  $B-N(\text{cone points})$  having zero slope over  $\partial N(\text{cone points})$ ; otherwise  $\eta = 1$ . So, for



$\eta \in \mathbb{Z}_2$  is actually an invariant of  $Q' \rightarrow B'$  i.e. either there exists a global section or there does not, irrespective of any fibred isomorphisms.



we can change  $\eta$  by altering the parameterization of the cone points e.g.



The consequence of all this is that the only crystals  $Q' \rightarrow B'$  with  $\partial|B'| = \emptyset$  and non-redundant  $\eta \in \mathbb{Z}_2$  are:





and to detect them, once we know the Seifert type, all we have to do is to establish the existence of a global section (so  $\eta = 0$ ) or its absence (so  $\eta = 1$ ): when we have



we can "ignore" the cone fibrations by choosing the obvious parameterization,

treating the singular fibres as if they are regular. In other words, we may think just in terms of the (two) non-orientable manifold  $S^1$ -bundles over  $\mathbb{R}P^2$ .

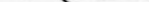
Similarly to Proposition (4.8.2), each of the above cases ( $\eta = 0$  versus  $\eta = 1$ ) is distinguished by having *either* a 1-fold global section or a connected 2-fold global section, the two being mutually exclusive. Thus, knowing what  $Q \rightarrow B$  is, by looking at partial 1-fold or 2-fold sections of it and seeing which global sections of  $Q' \rightarrow B'$  arise from them (by lifting to  $Q'$  and completing), we can decide on the nature of  $Q' \rightarrow B'$  i.e. whether it has  $\eta = 0$  or  $\eta = 1$ . Checking each case is routine but tedious, so we just list the results here with a brief explanation of their derivation. Remember: we have to look only at those  $Q \rightarrow B$  which (4.7) tells us lift to  $Q' \rightarrow B'$  of Seifert type  $N_n I$  or  $N_n II$ .

$Q' \longrightarrow B'$   
 (4.2.5)   $\longrightarrow$  

When  $Q \rightarrow B$  has only type 4 points ( $\triangleleft$ ), cones of slope  $\frac{1}{2}$  ( $\frac{1}{2}\triangleleft$ ),  $\infty$  or  $\infty$  then we can construct a 1-fold section lifting to another in  $Q' \rightarrow B'$ . Otherwise there is a 2-fold section of  $Q \rightarrow B$ , lifting to the same in  $Q' \rightarrow B'$  made up of two disjoint components, either of which is the required 1-section.



(4 8 6)  $\begin{matrix} \eta=1 \\ T_{3/2} \\ N_0 \Sigma \end{matrix}$   $\longrightarrow$

For these there is a 2-fold section of  $Q \rightarrow B$  lifting to a connected 2-fold section of  $Q' \rightarrow B'$

(42.7) 

Either there is a 1-section of  $Q \rightarrow B$  ( $q=0$  or slope  $9/5$ ) or else a 2-section splits to two 1-sections of  $Q \rightarrow B$  (the other two cases).


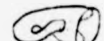
$$Q' \longrightarrow B'$$



(4.8.8)   $\xrightarrow[\tau_{3pe}]{\eta=1}$  

$$Q \longrightarrow B$$

  $\xrightarrow[\tau_{3pe}]{\eta=1}$  



The 2-section of  $Q \rightarrow B$  remains connected on lifting to  $Q' \rightarrow B'$

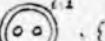

(4.8.9)   $\xrightarrow[\tau_{3pe}]{\eta=0}$  

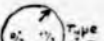

  $\xrightarrow[\tau_{3pe}]{\eta=0}$  

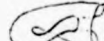

  $\xrightarrow[\tau_{3pe}]{\eta=0}$  

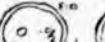

(Twice - see 4.4.1)

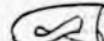

(4.8.10)   $\xrightarrow[\tau_{3pe}]{\eta=1}$  

  $\xrightarrow[\tau_{3pe}]{\eta=1}$  

  $\xrightarrow[\tau_{3pe}]{\eta=1}$  

(4.8.11)   $\xrightarrow[\tau_{3pe}]{\eta=0}$  

  $\xrightarrow[\tau_{3pe}]{\eta=0}$  

(4.8.12)   $\xrightarrow[\tau_{3pe}]{\eta=1}$  

  $\xrightarrow[\tau_{3pe}]{\eta=1}$  

(4.9) Now suppose that the generic fibre of  $Q \rightarrow B$  is  $mI$  and that we wish to construct a fibred double cover  $Q' \rightarrow B'$ , where  $\bar{\phi}: B' \rightarrow B$  is a double covering map of 2-orbifolds i.e. we have a commutative diagram:

$$\begin{array}{ccc} Q' & \xrightarrow{\quad \phi \quad} & Q \\ p'_* \downarrow & & \downarrow p \\ B' & \xrightarrow{\quad \bar{\phi} \quad} & B \end{array}$$

In terms of the fundamental groups we have a diagram of short exact sequences

$$\begin{array}{ccccc} \mathbb{Z} \times \mathbb{Z}_2 & \longrightarrow & G_Q & \xrightarrow{p_*} & \begin{array}{c} G_{B'} \\ \bar{\phi}_* \downarrow \\ G_B \\ \eta \downarrow \\ \mathbb{Z}_2 \end{array} \end{array}$$

where  $\eta: G_B \rightarrow G_B/G_B \cong \mathbb{Z}_2$  is the natural homomorphism and, of course,

$$\pi_1(mI) \cong \mathbb{Z} \times \mathbb{Z}_2 \quad (\text{see 0.5.1}).$$

Make the following definitions, just as for the case when the generic fibre was  $S^1$  (4.1) :

$$\begin{aligned} G_{Q'} &= p_*^{-1} \bar{\phi}_* (G_{B'}) \\ \phi_* &= \text{id}|_{G_{Q'}} : G_{Q'} \rightarrow G_Q \quad (\text{so } \phi_* \text{ is injective}) \\ p'_* &= \bar{\phi}_*^{-1} \cdot p_*|_{G_{Q'}} : G_{Q'} \rightarrow G_{B'} \quad (\text{so } p'_* \text{ is surjective}). \end{aligned}$$

These give us the following diagram of short exact sequences:

$$(4.9.1) \quad \begin{array}{ccccc} \mathbb{Z} \times \mathbb{Z}_2 & \longrightarrow & G_{Q'} & \xrightarrow{p'_*} & G_{B'} \\ \downarrow \cong & & \phi_* \downarrow & & \bar{\phi}_* \downarrow \\ \mathbb{Z} \times \mathbb{Z}_2 & \longrightarrow & G_Q & \xrightarrow{p_*} & G_B \\ & & \downarrow & & \downarrow \eta \\ & & G_Q/G_{Q'} & \xrightarrow[\cong]{\tau} & \mathbb{Z}_2 \end{array}$$

$\tau: G_Q/G_{Q'} \rightarrow \mathbb{Z}_2$  is defined by  $\tau(\alpha \cdot G_{Q'}) = \eta \circ p_*(\alpha)$  (and so is a well-defined isomorphism by Lemma 4.2.1).

Given a sufficiently small co-ordinate chart  $U \cong D^2/\Gamma$  of  $B$ , we know that  $p^{-1}(U)$  is one of the local fibre types 12 - 19 (see §1) and has the form

$$\begin{array}{ccc} (D^2 \times S^1)/(\mathbb{Z}_\alpha \times \mathbb{Z}_2) & \text{or} & (D^2 \times S^1)/(\Delta_{2\alpha} \times \mathbb{Z}_2) \\ \downarrow p & & \downarrow \\ D^2/\mathbb{Z}_\alpha & & D^2/\mathbb{Z}_\alpha \end{array}$$

where the extra  $\mathbb{Z}_2$ -action (not projecting to the base) is what makes the fibres into mI's. Just as when the generic fibres are  $S^1$ 's,  $\bar{\phi}^{-1}(U)$  consists either of a single connected component which is a double cover of  $U$  or of two disjoint copies of  $U$ . In the former case, with  $p^{-1}(U) \cong (D^2 \times S^1)/\Gamma$ , we must have

$$p'^{-1} \bar{\phi}^{-1}(U) \cong (D^2 \times S^1)/\Gamma',$$

where  $\Gamma' \leq \Gamma$  of index 2; this time we read off the required local fibre type from Table 3.2. In the latter case,  $p'^{-1} \bar{\phi}^{-1}(U)$  is isomorphic to two disjoint copies of

$$(D^2 \times S^1)/\Gamma \cong p^{-1}(U) .$$

In passing, note that this tells us that the generic fibre of  $Q' \rightarrow B'$  must also be  $m1$  and that the first case actually looks like

$$p'^{-1} \bar{\phi}^{-1}(U) \cong (D^2 \times S^1)/(\Lambda' \times \mathbb{Z}_2) ,$$

where  $G \cong \Lambda \times \mathbb{Z}_2$  ,  $|\Lambda:\Lambda'| = 2$  and  $\bar{\phi}^{-1}(U) = D^2/\Lambda'$  .

The classification of euclidean  $S^1$ -fibred 3-orbifolds with generic fibre  $m1$  (up to base-free fibred isomorphism) is given in (2.4); equivalence of two diagrams is simply by equality. It is clear that for a given cover  $\bar{\phi}:B' \rightarrow B$  the local information about  $Q' \rightarrow B'$  (giving us the type diagram) can be derived from  $Q \rightarrow B$  just as for the case with generic fibre  $S^1$  (4.2.3). It remains to determine:-

(4.10) Which components of  $\partial|B|$  are marked by '\*' s.

(4.11) If no '\*' s are marked in (4.10) and there are no type 14 cone points, which Seifert symbol  $Q' \rightarrow B'$  possesses; the list of permissible Seifert symbols is just as for the case of generic fibre  $S^1$  - see (2.4).

(4.10) The argument proceeds along the lines of the case when the generic fibre is  $S^1$  . 16 of the 17 2-crystals  $B'$  have either  $\partial|B'| = \emptyset$  (so there can be no '\*' s) or  $\partial|B'|$  is a single component, in which case the presence or absence of a '\*' depends exactly on whether there is an odd or even number of type 14 fibres in  $Q' \rightarrow B'$  . The only work has to be done in the case when  $|B'| = \text{annulus}$ . As before, we consider the possibilities for

$\bar{\phi}: B' \rightarrow B$ , the arguments being substantially the same:

$$(4.10.1) \quad \begin{array}{ccc} \text{Diagram of } B' & \xrightarrow{\bar{\phi}} & \text{Diagram of } B \end{array}$$

The diagram on the left shows two concentric circles, with the outer one labeled  $B'$ . The diagram on the right shows a circle with two points inside, labeled  $i$  and  $j$ , and the outer boundary labeled  $B$ .

There are '\*' s on (both components of)  $\partial|B'|$  exactly when there is a '\*' on  $\partial|B|$ .


$$(4.10.2) \quad \begin{array}{ccc} \text{Diagram of } B' & \xrightarrow{\bar{\phi}} & \text{Diagram of } B \end{array}$$

The diagram on the left shows two concentric circles, with the outer one labeled  $B'$ . The diagram on the right shows a square with vertices labeled 2, 2, 2, 2, and the outer boundary labeled  $B$ .

Depending on along which pair of sides we unfold,  $B'$  has '\*' s exactly when one side is of type 15 and the other is of type 16 i.e.

$$\text{one } \begin{array}{c} \text{Diagram of } B' \\ \text{with } \partial|B'| \end{array} \quad \text{and one } \begin{array}{c} \text{Diagram of } B' \\ \text{with } \partial|B'| \end{array}$$

The first diagram shows a semi-circle with a horizontal line segment below it, labeled 1. The second diagram shows a semi-circle with a horizontal line segment below it, labeled R.

(4.11) Assume that  $Q' \rightarrow B'$  is not assigned any '\*' s in (4.10) and that there are no type 14 fibres i.e. no cone points marked . Checking back to the list of possible Seifert types (2.3), we again have only the following cases to deal with:

$$B' \cong \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

The diagrams show four different topological surfaces: a torus, a Möbius band, a Klein bottle, and a sphere with a point.

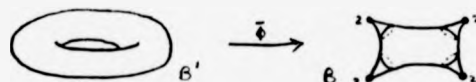
Therefore the covers of concern to us are again those of (4.7). The manner of dealing with them is exactly as we did in (4.7.1) to (4.7.9), except that now we are looking for Möbius bands or annuli (as opposed to Klein bottles or tori).


$$(4.11.1) \quad \begin{array}{ccc} \text{Diagram of } B' & \xrightarrow{\bar{\phi}} & \text{Diagram of } B \end{array}$$

The diagram on the left shows a torus with a horizontal line segment, labeled  $B'$ . The diagram on the right shows a torus with a horizontal line segment, labeled  $B$ .

$Q \rightarrow B$  is of Seifert type  $\underline{0} \Rightarrow Q' \rightarrow B'$  is of Seifert type  $\underline{0}$





$Q \rightarrow B$  is of Seifert type  $N_0 \Rightarrow Q' \rightarrow B'$  is of Seifert type  $N_0$ .

(4.11.2) 







$Q' \rightarrow B'$  is of Seifert type  $N_0$  exactly when  $Q \rightarrow B$  has type diagram: ; otherwise it has Seifert type 0.

(4.11.3) 

If  $\omega \subset B$  is a closed path as shown (lifting to  $\omega' \subset B'$ ) then the following table determines the Seifert type of  $Q' \rightarrow B'$ . (Take care that  $\omega$  meets the appropriate side of  $B$ , i.e. the "unwrapped" one.)

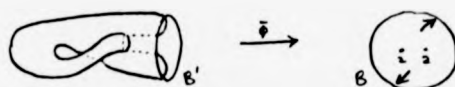
	cone point of $Q \rightarrow B$ is type 13 i.e. 	cone point of $Q \rightarrow B$ is type 14 i.e. 
$\omega$ meets 	$Q' \rightarrow B'$ has Seifert type $N_n I$	$Q' \rightarrow B'$ has Seifert type <u>0</u>
$\omega$ meets 	$Q' \rightarrow B'$ has Seifert type <u>0</u>	$Q' \rightarrow B'$ has Seifert type $N_n I$

(4.11.4) 

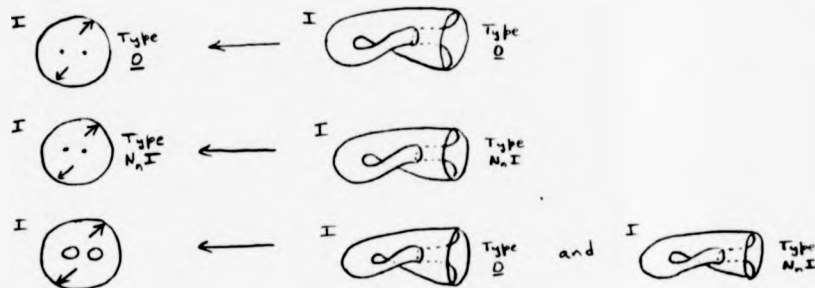
If $Q \rightarrow B$ is:	then $Q' \rightarrow B'$ is of Seifert type:
 or 	$N_n II$
 or 	$N_n I$
 or 	<u>0</u>



(4.11.5)



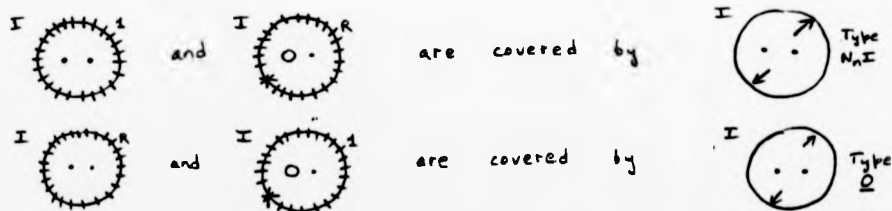
There are only three possibilities for  $Q \rightarrow B$ . We list their covers which have base a Klein bottle :



(4.11.6)

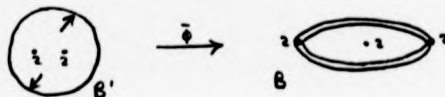


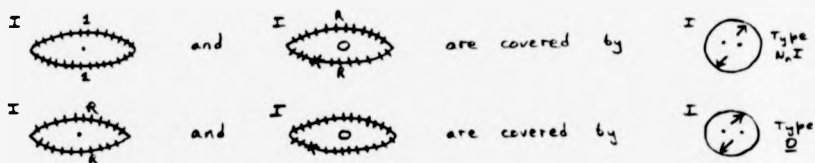
We have :



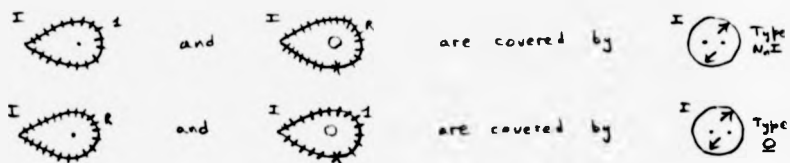
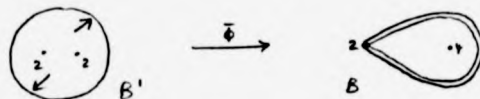
Note : the case of two type 14 cone points in  $Q \rightarrow B$  does not concern us because two copies of one of these fibrations would lift to  $Q' \rightarrow B'$ , which would then not have a Seifert type assigned to it. Thus when there is one fibration of each type we must be unwinding about the type 14 fibre.

(4.11.7)





(4.11.8)



§5. Determination of S-Covers.

In this chapter we deal with S-covers: given an  $S^1$ -fibred 3-crystal  $Q \rightarrow B$ , we wish to construct all of its fibred double covers  $Q' \rightarrow B'$  for which  $\bar{\phi}: B' \rightarrow B$ , the projection of the covering map to the base spaces, is a homeomorphism (as opposed to a double covering). Thinking of  $\bar{\phi}$  as the identity map, for a given  $Q \xrightarrow{p} B$  we wish to construct  $Q' \xrightarrow{p'} B$  and  $\phi: Q' \rightarrow Q$  such that the following diagram commutes:

$$(5.0) \quad \begin{array}{ccc} Q' & \xrightarrow{\phi} & Q \\ & \searrow p' & \downarrow p \\ & & B \end{array}$$

i.e.  $\phi: Q' \rightarrow Q$  projects to the identity map on  $B$ . Notice that S-covers are fundamentally different from D-covers: whereas in (4.2.2) we saw that for a given double cover  $\bar{\phi}: B' \rightarrow B$  there is a unique double cover  $Q' \rightarrow B'$  of  $Q \rightarrow B$ , here there may be more than one cover of the form  $Q' \rightarrow B$ , or none at all. For instance, when the generic fibre of both  $Q$  and  $Q'$  is  $S^1$  we have the following short exact sequences of fundamental groups, where the solid arrows are given and we must determine the broken ones:

$$\begin{array}{ccccc} \mathbb{Z}\mathbb{Z} & \xrightarrow{\quad} & G_{Q'} & \xrightarrow{p'_*} & G_B \\ \downarrow & & \downarrow \phi_* & & \downarrow \cong \\ \mathbb{Z} & \xrightarrow{\quad} & G_Q & \xrightarrow{p_*} & G_B \\ \downarrow & & \downarrow & & \\ \mathbb{Z}/\mathbb{Z}\mathbb{Z} & \xrightarrow{\sim} & G_Q/G_{Q'} & & \end{array}$$

It is clear that this need not be possible: every index 2 subgroup  $G_{Q'}$  of  $G_Q$  may correspond to an index 2 subgroup  $G_{B'}$  of  $G_B$ . Alternatively, two distinct subgroups  $G_{Q'}$  of  $G_Q$  may lift to the same subgroup of  $\mathbb{Z}$ .

There are three distinct cases depending on the generic fibres of  $Q'$  and  $Q$  respectively, viz:

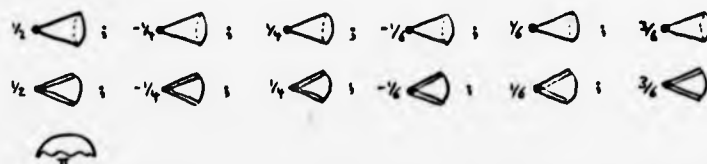
$$S^1 \rightarrow S^1 \quad (5.1)$$

$$mI \rightarrow mI \quad (5.2)$$

$$S^1 \rightarrow mI \quad (5.3)$$

(5.1)  $Q \xrightarrow{p} B$  has generic fibre  $S^1$  (and hence so does  $Q' \xrightarrow{p'} B$ )

First note that the local structure of  $Q \rightarrow B$  imposes conditions on whether it has any  $S$ -covers at all, since certain local fibre types don't possess them: namely, type 6 fibrations and those of type 1 and type 3 with slope  $\beta/\alpha$  such that  $\beta(1+\alpha)$  is odd i.e.  $\beta$  is odd and  $\alpha$  is even (where  $-\alpha/2 < \beta \leq \alpha/2$ ) - see Table 3.3. This means that if any of the following occur in the type diagram for  $Q \rightarrow B$  then we need go no further, since all the fibred double covers were provided by the  $D$ -covers of §4 :



Assuming that none of these occur, we can try to form a double cover  $Q' \rightarrow B$  of  $Q \rightarrow B$ . It is obvious how to do this: first cover  $B$  by a

(finite) collection of open sets  $\{U_i\}$  such that each  $W_i = p^{-1}(U_i)$  is one of the local fibre types of  $\pi$ . Then, starting with one of these ( $W_1$ , say), take a fibred double cover  $W'_1$ , also with base  $U_1$ , as per Table 3.3 - there may be more than one possibility for  $W'_1$  here. Having chosen  $W'_1$ , extend the process over all the  $U_i$ 's so that everything agrees on the overlaps. If it is possible to complete the process in a consistent fashion then we have built the local structure of a candidate for  $Q' \rightarrow B$ .

Unfortunately, because the local structure does not determine  $Q' \rightarrow B$  completely, or even ensure that there is a (global) double cover with the given local structure, in practice we must tread more carefully. The best way to understand this is to divide into separate cases depending on the nature of the underlying space of  $B$ .

(5.1.1)  $|B| \cong \text{disc}$  (so  $\partial|B|$  has a single component  $C_1$ )

$B$  is one of the following:



Let  $N(C_1)$  be a closed regular neighbourhood of the mirror cycle  $C_1$  in  $B$ ,  $B_0 = B - N(C_1)$  and  $Q_0 = p^{-1}(B_0)$ . Any singular points in  $B_0$  are cone points; assuming that none of the fibrations over them occurs in the above list of those not having  $S$ -covers, we can form at least one fibred double cover  $Q'_0 \rightarrow B_0$  of  $Q_0 \rightarrow B_0$ . Whenever more than one such cover is possible consider each case individually, and note that when  $B$  has two cone points any combination of coverings for the two local fibre types must work.

Having chosen our covering,  $\partial\overline{Q}_0^T$  is either a torus or a Klein bottle, fibred by orientation-preserving circles. Now we must construct the cover  $p'^{-1}(N(C_1)) \rightarrow N(C_1)$  of  $p^{-1}(N(C_1)) \rightarrow N(C_1)$ . The idea is to start at some  $x_0 \in C_1$ , take a fibred double cover of the local fibre type there (over the same base) and then traverse  $C_1$  taking double covers en route, with everything agreeing on the overlaps; finally, we ensure that on returning to  $x_0$  the double covering there agrees with the one we started with. For this reason it is best to start at a type 2 fibration if possible, since (over a fixed base) these can only be double covered by themselves and so we avoid unnecessary cases which end up failing to agree over  $x_0$ .

Unfortunately, this is not the whole story: there are various choices to be made and, when  $p^{-1}(C_1)$  is a Montesinos cycle, not only must we determine the parity of its double cover (which is necessarily also a Montesinos cycle), but there is a restriction on the above process which must be taken into account.


First, see that the weak type diagram is unaltered on taking double covers of this kind. In other words, fibres of types 1, 2, 3 and 4 are each covered by their own fibre type, and those of type 5 are covered by types 5 or 6. In particular,  $Q' \rightarrow B$  is orientable if and only if  $Q \rightarrow B$  is, since  $|B| \cong D^2$  and so we don't assign a Seifert type in (2.3(ii')).

Suppose that  $p^{-1}(C_1)$  is *not* a Montesinos cycle i.e. there are non-orientable fibre types over at least part of  $C_1$ ; then the same is

true of  $p'^{-1}(C_1)$ . The only choice involved in constructing a cover of  $p^{-1}(N(C_1))$  is whether to make  $\partial p'^{-1}(N(C_1))$  a torus or a Klein bottle; but the decision is forced by  $\partial \bar{Q}_0^+$ . "Twists" of  $p'^{-1}(C_1)$  do not trouble us because we do not have a Montesinos cycle. Nor do we have to worry about how  $\partial p'^{-1}(N(C_1))$  is glued to  $\partial \bar{Q}_0^+$ , since all ways are equivalent by (2.3). Of course, there may be more than one (non fibred-isomorphic) cover, and more than one  $Q'_0 \rightarrow B_0$ , and we must take every possible combination in producing the list of possible  $Q' \rightarrow B$ .


When  $p^{-1}(C_1)$  is a Montesinos cycle, life becomes more complicated. The following example illustrates the additional restrictions which come into play:

(5.1.2) Example: Suppose that

$$p^{-1}(N(C_1)) \cong \frac{1}{2} \text{ (annulus) } \quad \text{i.e.} \quad \text{(torus)}$$


(note that  $p^{-1}(C_1) \cong \text{annulus}$ ; were we to have  $\epsilon = 1$  it would be a Möbius band).

To form a double cover of this, start at  $x_0$  and go clockwise around  $C_1$ , taking double covers. On reaching the first dihedral point we have a choice as to its covering. Suppose we take this to be a type 3 fibre of slope  $1/2$  (we could have chosen slope  $0/2$ ) i.e.

$$\frac{1}{2} \text{ (triangle) } + \frac{0}{2} \text{ (triangle) }, \text{ so } p'^{-1}(C_1) \text{ looks like: } \text{(C-shape with dashed lines)}$$


Then on one side of the corner we are unwrapping  $p^{-1}(C_1)$  about  $\gamma$ , say, and on the other side about  $\delta$ ; the act of covering by a fibre of slope

$1/2$  means that we have "switched tracks" when unfolding the singular set of  $Q$ . Thus when we come to the last type 3 fibre before returning to  $x_0$  we are restricted in our choice of covering to one which unwraps in a suitable fashion: in this example we must again take

$$1/2 \begin{array}{c} \diagup \\ \diagdown \end{array} \rightarrow 0/2 \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \text{and not} \quad 0/2 \begin{array}{c} \diagup \\ \diagdown \end{array} \rightarrow 0/2 \begin{array}{c} \diagup \\ \diagdown \end{array}$$

Had we started with:  $0/2 \begin{array}{c} \diagup \\ \diagdown \end{array} \xrightarrow{\epsilon=1} 0/2 \begin{array}{c} \diagup \\ \diagdown \end{array}$  (so  $p^{-1}(C_1) \cong \text{Möbius band}$ ),

then, if our first covering were by a fibre of slope  $1/2$ , the second would have to be by one of slope  $0/2$ : the  $\epsilon = 1$  introduces a half-twist into the singular set of  $p^{-1}N(C_1)$ . //

So, if we present  $p^{-1}(C_1)$  in standard form, i.e.  $-1/2 < \beta/\alpha \leq 1/2$  for all slopes  $\beta/\alpha$ , then we require:

$$\Sigma(\text{no. of "switches"}) + \epsilon(C_1) \equiv 0 \pmod{2}.$$

The covers of the type 3 fibres have the form:

$$\beta/\alpha \begin{array}{c} \diagup \\ \diagdown \end{array} \leftarrow \frac{\frac{1}{2}(\beta + k\alpha)}{\alpha} \begin{array}{c} \diagup \\ \diagdown \end{array} \quad (k \in \mathbb{Z})$$

and it is an easy exercise to show that we get a "switch" when  $k = \pm 1$  and not when  $k = 0$ , leading to:

$$(5.1.3) \quad \sum_{i=1}^n k_i + \epsilon(C_1) \equiv 0 \pmod{2},$$

where  $n$  is the number of corners in  $C_1$ .



The covers for which  $k = \pm 1$  are :


$$\frac{0}{2} + \frac{1}{2}, \frac{1}{3} + \frac{-1}{3}, \frac{-1}{3} + \frac{1}{3}, \frac{0}{4} + \frac{2}{4}, \frac{2}{4} + \frac{-1}{4}, \frac{0}{6} + \frac{3}{6}, \frac{2}{6} + \frac{-2}{6}, \frac{-2}{6} + \frac{2}{6} ;$$

all other (euclidean) ones have  $k = 0$  and make no contribution to the sum.

When constructing the cover of a Montesinos cycle in this way we are therefore restricted by (5.1.3). Also, just as before, we must ensure that the boundaries match i.e. that

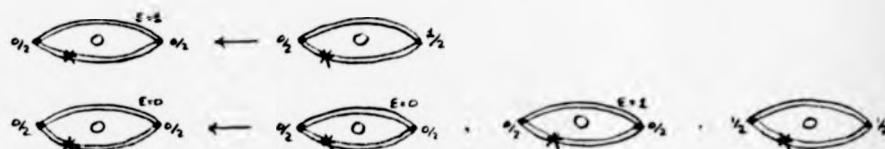
$$\partial(\overline{p'^{-1}N(C_1)}) \cong \partial \overline{Q_0} .$$

Having constructed  $p'^{-1}N(C_1)$ , we must determine the parity of  $p'^{-1}(C_1)$ . A half-twist in it projects to a full twist of  $p^{-1}(C_1)$ . When  $Q \rightarrow B$  is non-orientable this full twist leaves  $Q \rightarrow B$  unaltered (up to fibred isomorphism) - see (2.3(iii)) - and so both cases ( $\epsilon = 0$  and  $\epsilon = 1$ ) work for  $Q' \rightarrow B$ . Note, however, that these can be fibred-isomorphic to one another and so reduce to a single case in practice e.g.:

$$70 \text{ Fddd } 0/2 \text{  1/2 \text{ (} \epsilon = 0 \text{ and } \epsilon = 1 \text{ cases are equivalent} \\ \text{- see (2.5.3))}$$

When  $Q \rightarrow B$  is orientable, a full twist in  $p^{-1}(C_1)$  (with everything else unchanged) alters the Euler number  $e_0(p) \in \mathbb{Q}$  by  $\pm 1$ . Since  $Q \rightarrow B$  has  $e_0(p) = 0$ , there is a unique parity  $\epsilon(C_1)$  for  $p'^{-1}(C_1)$ ; this is confirmed by the fact that, for an orientable  $Q' \rightarrow B$  with  $|B|$  a disc (as we have here), the parity of the Montesinos cycle in  $Q'$  is completely determined by the local structure.

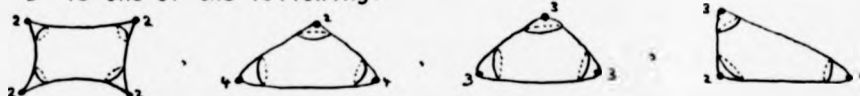
Examples:



Since no other invariants occur for the case  $|B| \cong \text{disc}$ , we are finished.

$$(5.1.4) \quad |B| \cong S^2 \quad (\text{so } \partial|B| = \emptyset)$$

$B$  is one of the following:



First assume that  $Q \rightarrow B$  is non-orientable. Since  $|B|$  is simply-connected and  $\partial|B| = \emptyset$ , it follows that  $Q$  has a type 4 fibration over a cone point  $x_0 \in B$ . Just as in (5.1.1), we construct a fibred double cover  $Q_0^+ \rightarrow B_0$  of  $Q_0 \rightarrow B_0$  by piecing together local covers, where  $B_0$  is now the complement of  $N(x_0)$ , a neighbourhood of  $x_0$  in  $B$ . Since  $p^{-1}(N(x_0))$  is a type 4 fibration, and hence so is any double cover  $p'^{-1}(N(x_0))$  of it, we have that  $\partial Q_0^+$  is a Klein bottle fibred by orientation-preserving circles, on which the covering involution is just a rotation through half their length. By gluing  $\partial Q_0^+$  to  $(p'^{-1}(N(x_0)))$ , preserving fibres, we automatically have a covering of  $Q \rightarrow B$ . The classification of non-orientable  $S^1$ -fibred orbifolds confirms that there is no ambiguity in the gluing - all the objects so obtained are fibred-isomorphic to one another by (2.3).

Now suppose that  $Q \rightarrow B$  is orientable. Then we can cover it in the obvious way, subject to the restriction:

$$(5.1.5) \quad \sum_{i=1}^n \beta_i / \alpha_i + \sum_{i=1}^n k_i \equiv 0 \pmod{2}.$$

$n$  is the number of cone points and  $k_i$  is related to the covering of each cone point of slope  $\beta_i/\alpha_i$  by

$$\frac{\beta_i}{\alpha_i} \triangleleft \frac{\frac{1}{2}(\beta_i + k_i \alpha_i)}{\alpha_i} \triangleleft$$

This is enough to ensure that  $Q' \rightarrow B$  is euclidean i.e. that

$$\sum_{i=1}^n \frac{\frac{1}{2}(\beta_i + k_i \alpha_i)}{\alpha_i} \in \mathbb{Z}$$

(and so  $e_0(p') = 0$ ).

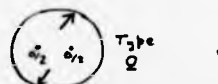
We can omit the cases for which  $B$  is one of the following:



because any crystal which fibres over one of these three has an alternative fibration over one of the other eleven 2-crystals. This leaves only two others to deal with:



(5.1.6) Taking the first of these, the only orientable case which admits a double cover of this type is





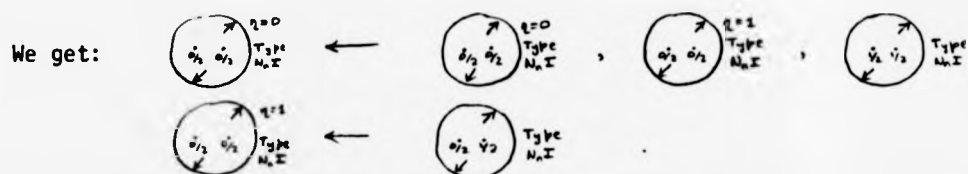
since the other possibility contains local fibrations which do not have S-covers. It is straightforward to see that this is covered by:



Now suppose that  $Q \rightarrow B$  is non-orientable. Then we have:



(Since  fibred differently is just , we will not deal with it here.)



The first two are easy to see: just ignore the singular fibres and treat them as fibred manifolds. The third has a connected 2-fold section which projects to a 1-section i.e. the  $\eta = 0$  case. For the last, the right hand side admits a connected 4-fold section projecting to a connected 2-fold section, giving us  $\eta = 1$ .

(5.1.7) The three cases are:



Regardless of whether  $Q' \rightarrow B$  admits a connected 1-fold or 2-fold section, this must project to a 1-fold section of  $Q \rightarrow B$ .

(5.2)  $Q \xrightarrow{P} B$  and  $Q' \xrightarrow{P'} B$  both have generic fibre  $mI$ .

If  $Q \rightarrow B$  has any local fibrations of types 14 or 16 i.e.



then no such covers exist (see Table 3.5); so suppose that it does not.



(5.2.1)  $|B| \cong \text{disc}$  Let  $C_1 = \partial|B|$  as before, and  $B_0 = B - N(C_1)$  be the complement of a closed neighbourhood of  $C_1$  in  $B$ . Then we can form a double cover  $Q_0 \rightarrow B_0$  of  $Q_0 \rightarrow B_0$  simply by piecing together local double covers. Note that whilst  $|\partial Q_0|$  is an annulus,  $|\partial Q_0|$  can be an annulus or a Möbius band.

Now we construct the double cover of  $p^{-1}N(C_1)$ , again by piecing together locally in a consistent fashion, with the final choice of how to glue the "ends" together (over some  $x_0 \in C_1$ , say) determined by whether  $|\partial p^{-1}N(C_1)|$  is required to be an annulus or a Möbius band to match  $\partial|\overline{Q}_0|$ .

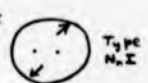

(5.2.2)  $|B| \cong S^2$  Just as in (5.2.1), we can piece together any combination of local double covers that we wish, subject to the condition that gluing in the last one makes sense i.e. the boundaries must match. This is exactly the condition that  $n_{14} \equiv 0 \pmod{2}$  i.e. that the number of type 14 fibrations in  $Q' \rightarrow B$  (the boundaries of neighbourhoods of which are Möbius bands) is even; this is required anyway, in order that the diagram for  $Q' \rightarrow B$  make sense.

(5.2.3)  $B \cong$   or 


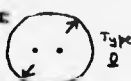
First, see that the following two crystals:

  $T_{14} = \frac{1}{2}$  and   $T_{14} = \frac{1}{2}$



cannot have double covers of this type, because for both of them there is a simple closed curve  $K \subset B$  such that  $|p^{-1}(K)|$  is a Möbius band (fibred by  $m_1$ 's), which cannot be double covered in the required way.

This leaves   $T_{14} = \frac{1}{2}$  and   $T_{14} = \frac{1}{2}$

which, it is easy to see, are covered respectively by:

  $T_{14} = \frac{1}{2}$  and   $T_{14} = \frac{1}{2}$

and



  $T_{14} = \frac{1}{2}$  and   $T_{14} = \frac{1}{2}$

Now we construct the double cover of  $p^{-1}N(C_1)$ , again by piecing together locally in a consistent fashion, with the final choice of how to glue the "ends" together (over some  $x_0 \in C_1$ , say) determined by whether  $|\partial p^{-1}N(C_1)|$  is required to be an annulus or a Möbius band to match  $|\partial Q_0|$ .



(5.2.2)  $|B| \cong S^2$  Just as in (5.2.1), we can piece together any combination of local double covers that we wish, subject to the condition that gluing in the last one makes sense i.e. the boundaries must match. This is exactly the condition that  $\eta_{14} \equiv 0 \pmod{2}$  i.e. that the number of type 14 fibrations in  $Q' \rightarrow B$  (the boundaries of neighbourhoods of which are Möbius bands) is even; this is required anyway, in order that the diagram for  $Q' \rightarrow B$  make sense.

(5.2.3)  $B \cong$   or 



First, see that the following two crystals:

 and 



cannot have double covers of this type, because for both of them there is a simple closed curve  $K \subset B$  such that  $|p^{-1}(K)|$  is a Möbius band (fibred by  $m_i$ 's), which cannot be double covered in the required way.

This leaves  and 

which, it is easy to see, are covered respectively by:

 and 

and

 and 

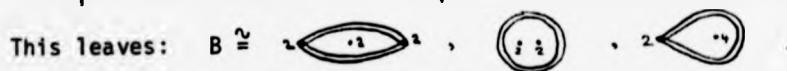
(5.3)  $Q \xrightarrow{p} B$  has generic fibre  $m\mathbb{I}$  and  $Q' \xrightarrow{p'} B$  has generic fibre  $S^1$ .


(5.3.1)  $|B| \cong \text{disc}$ . As usual, let  $C_1 = \partial|B|$ ,  $B_0 = B - N(C_1)$ ,  $Q_0 = p^{-1}(B_0)$  and cover  $Q_0 \rightarrow B_0$  by  $Q'_0 \rightarrow B_0$ .  $\partial Q'_0$  is either a torus or a Klein bottle. Construct a cover of  $p^{-1}N(C_1)$  by beginning at some  $x_0 \in C_1$ . Using Table 3.4, take a fibred double cover of the local fibre type there, and then continue to take such covers (in a consistent fashion) going around  $C_1$  until returning to  $x_0$ , ensuring that the final fibre type constructed there agrees with the initial one. If  $p'^{-1}(C_1)$  is not a Montesinos cycle then we are done, since  $p'^{-1}N(C_1)$  is unique up to fibred isomorphism, once we have chosen the boundary to agree with  $\partial Q'_0$ . We can glue the two boundaries together in any (fibre-preserving) way that we like, since all such ways are equivalent by the classification of (non-orientable) orbifolds with generic fibre  $S^1$ .

If  $p'^{-1}(C_1)$  is a Montesinos cycle, we must determine its parity. If  $Q' \rightarrow B$  is orientable, so in particular if

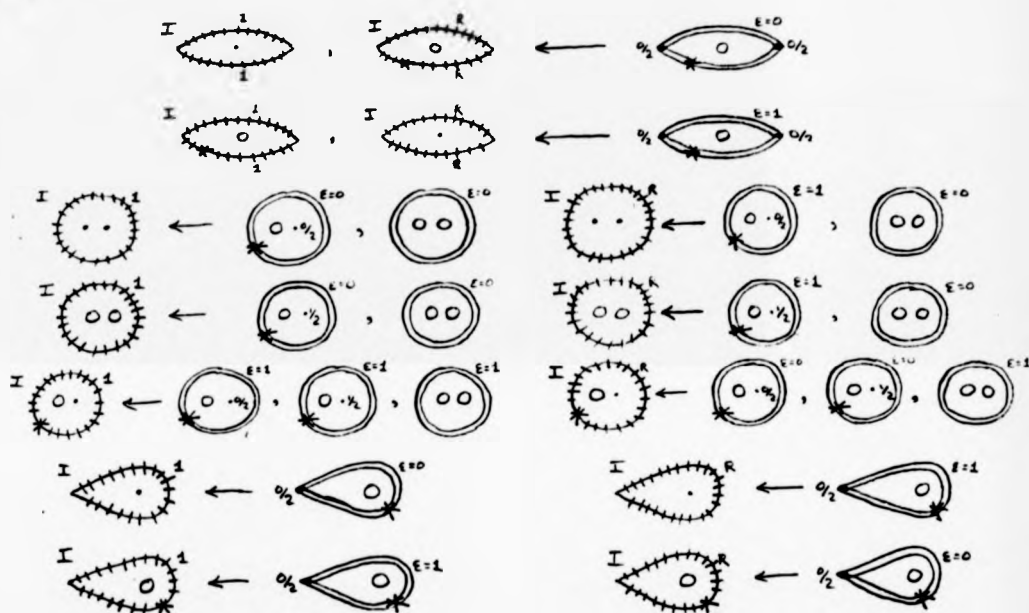


then  $\epsilon'(C_1)$ , the parity of  $p'^{-1}(C_1)$ , is determined by the local structure.



We list the results below, omitting all those cases which are completely determined by their local structure. This list is obtained by noting that when  $p^{-1}(C_1)$  is fibred by type 15 points i.e. , then these lie as an annulus or a Möbius band when  $\partial Q_0$  is an annulus or Möbius band respectively, and lift as a copy of themselves to form  $p'^{-1}(C_1)$ . When

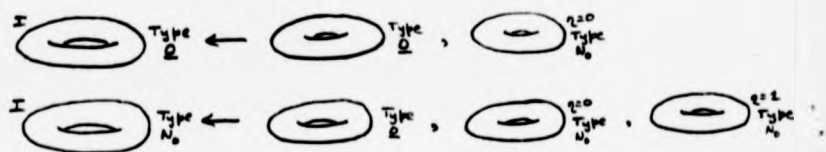
$p^{-1}(C_1)$  is fibred by type 16 points i.e.  $\mathbb{I}$ , these always lie as an annulus; whether  $p^{-1}(C_1)$  is an annulus or a Möbius band depends on  $\partial\overline{Q_0}$  and  $\partial\overline{Q_0^*}$ .



(5.3.2)  $|B| \approx S^2$  Just as in (5.1.4), any cover that we can construct by piecing together the local double covers must work, subject only to the provisos that it makes sense (i.e. gluing in the last boundary component is possible) and, when  $Q' \rightarrow B$  so constructed is orientable, that it is also euclidean i.e. that  $e_0(p') = 0$ .

(5.3.3)  $B \approx$

We have:





If we view  $mI \times S^1 \times S^1$  as being the product of an  $mI$ -fibred annulus with a circle, then the first two covers are the result of doubling this annulus to form a torus and a Klein bottle respectively. This is why the  $n = 1$  case does not double cover it; the fact that  $mI \times S^1 \times S^1$  has two distinct boundary components ensures that either of them lifts to a section of the double covering.

Thinking of the second  $Q \rightarrow B$  as the product of an  $mI$ -fibred Möbius band with a circle, we can double this Möbius band to either a torus or a Klein bottle. For the latter, both the cases  $n = 0$  and  $n = 1$  may occur, depending on whether one Möbius band half of the Klein bottle returns to itself or not when the product with the circle is taken.

(5.3.4)  $B \cong \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$

We have:

$$\begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array}$$

$$\begin{array}{c} I \\ \circlearrowright \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowright \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowright \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowright \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowright \end{array}$$

(There is no need to consider the third possibility for  $Q \rightarrow B$ , namely  $\begin{array}{c} \circ \circ \end{array}$ , since this has already been dealt with as  $\begin{array}{c} \circ \circ \end{array}$ .)

Letting  $\omega$  lie in  $B$  as shown:  $\begin{array}{c} \omega \\ \circlearrowleft \end{array}$ ,

for the first  $Q \rightarrow B$ ,  $p^{-1}(\omega)$  is a Möbius band, and for the second an annulus (both  $mI$ -fibred). Either can lift to the  $S^1$ -fibred Klein bottle required to make  $Q' \rightarrow B$  of Seifert type  $\underline{0}$ . Now suppose that they lift to a torus, so that  $Q' \rightarrow B$  has Seifert type  $N_n I$ . Since

$$\begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array} \xleftarrow{T_{ypc}} \begin{array}{c} I \\ \circlearrowleft \end{array}$$

has underlying space  $\mathbb{R}P^2 \times I$ ,  $\mathbb{R}P^2 \times \{0\}$ , say, must lift to a section of  $Q' \rightarrow B$ , giving us  $\eta = 0$ . Whereas,



must have underlying space with boundary  $S^2$ , since it is orientable. This lifts to a connected 2-fold section giving us  $\eta = 1$ . The last cover in both cases is determined entirely by its local structure.

§6. Non-Fibred 3-crystals.

In §3 we listed the first 194 crystallographic space groups together with their corresponding  $S^1$ -fibred 3-orbifolds. There are a further 36 groups (up to orientation-preserving affine homeomorphism) which fail to preserve any direction in  $\mathbb{R}^3$  and consequently have as quotient a 3-crystal which cannot be  $S^1$ -fibred. These are what crystallographers refer to as the *cubic* system of groups, so called because their point groups (of orders 12, 24 or 48) are symmetries of a cube. In particular they are characterized by having an axis of order 3 which cyclically permutes the standard perpendicular axes  $O_x$ ,  $O_y$  and  $O_z$ . There is a further subdivision into five classes, depending on the exact nature of the point group  $G_\infty$ , as follows:

$G_\infty$	Number in Class
23 ( $T_{12}$ )	5
m3 ( $\bar{T}_{24}$ )	7
432 ( $O_{24}$ )	8
$\bar{4}3m$ ( $\bar{T}_{24}$ )	6
m3m ( $\bar{O}_{48}$ )	10
	<u>36</u>

(the International Notation for  $G_\infty$  is followed here by the Schönflies notation, giving the order of the point group, in parentheses ).

We now explain the nature of each class of groups; remember that, although the action is considered to be *at a point* in the following (so, for instance, all the axes of rotation intersect one another), this may not necessarily be so in the space group  $G$  - reflections and rotations may be accompanied by a shift in a given direction.

$23$ , or  $T_{12}$ , denotes the *tetrahedral group*, and consists of all the rotations which preserve a regular tetrahedron; thus it has three perpendicular axes  $(O_x, O_y, O_z)$  of order 2 (joining the mid-points of edges) and four axes of order 3 (passing through the vertices), each of which cyclically permutes the order 2 axes; it is easily seen to have order 12, and is orientation-preserving i.e. is a subgroup of  $SO(3)$ .

Introducing an involution to the tetrahedral group gives us  $m3$  (or  $\bar{T}_{24}$ ). This now contains three perpendicular reflection planes and so is no longer orientation-preserving, lying instead in  $O(3)$ .

$432$ , or  $O_{24}$ , consists of all the rotations which preserve a cube.  $O_x, O_y$  and  $O_z$  are now of order 4 and the axes of order 3 are as before. Thus the tetrahedral group is immediately seen to be a proper subgroup. There are also six axes of order 2, joining mid-points of opposite edges of the cube. This is known as the *cubic* or *octahedral* group, is again orientable, and has order 24.

Writing the cubic group  $O_{24}$  as  $G$  and the tetrahedral group  $T_{12}$  as  $H$ , we have observed that  $H \triangleleft G$  (since  $|G:H| = 2$ ). Thus we can write

$$G = H + rH,$$

for some rotation  $r \in G-H$ , and can define a new group

$$G' = H + irH,$$

where  $i$  is inversion. We denote  $G'$  by  $\bar{4}3m$  or  $\bar{T}_{24}$ . It has the same order 4 and order 3 axes as the cubic group, but now contains reflections in six planes each containing one of  $O_x, O_y$  and  $O_z$  and

lying at  $45^\circ$  to the other two. It has order 24 and is not orientable.

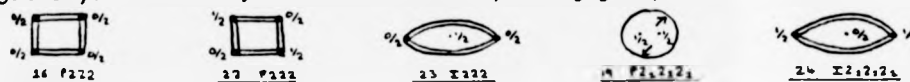
Finally we have  $m\bar{3}m$  (or  $\bar{O}_{48}$ ), which is obtained from the cubic group  $O_{24}$  just as  $\bar{T}_{24}$  was obtained from the tetrahedral group  $T_{12}$ , by introducing an involution. This has order 48 and is non-orientable.

Of course, for the sake of consistency we should seek to derive all the crystallographic space groups of the cubic system by topological methods i.e. determining the 3-crystals which are the quotient of their actions on  $\mathbb{R}^3$ . This is done in [B + S] for the first of the classes i.e. those with  $G_\infty \cong T_{12}$ , but the remainder were left to the reader. Before proceeding, we take the opportunity of giving a complete list here. Although it is possible to derive this simply by looking at group actions on the appropriate orbifolds, it seems that the only way to decide whether a given orbifold corresponds to a group whose point group is, say,  $\bar{T}_{24}$  or  $\bar{T}_{24}$ , or to decide exactly which group a given crystal corresponds to, is to determine the crystals from scratch, starting with a representation of the group acting on  $\mathbb{R}^3$ . This is exactly how the crystals of the present list had their groups assigned to them. The orientable crystals herein were first listed in [Du 1].

(6.1)  $G_\infty \cong T_{12}$  [B + S:Y-51] If  $D_4$  denotes the dihedral group of order 4, generated by  $\pi$ -rotations about  $O_x$ ,  $O_y$  and  $O_z$ , then we have  $D_4 \triangleleft T_{12}$ , giving the following short exact sequences:

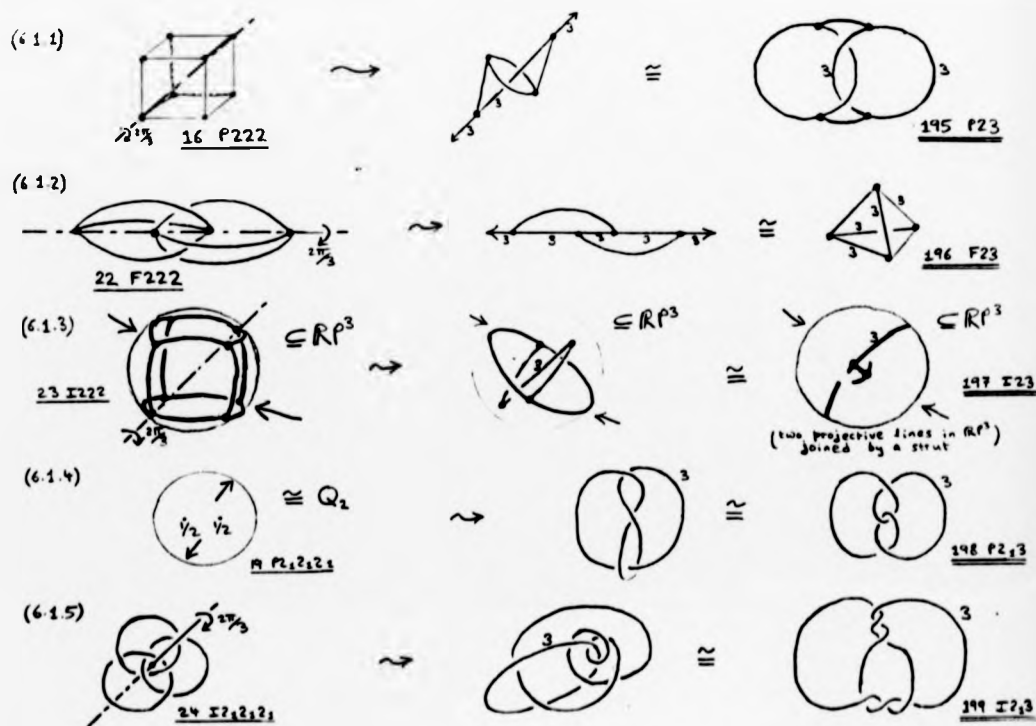
$$\begin{array}{ccccccc} 0 & \longrightarrow & G_0 & \longrightarrow & G & \xrightarrow{h} & T_{12} \longrightarrow 1 \\ & & \cong \parallel & & \vee & & \nabla \\ 0 & \longrightarrow & G_0 & \longrightarrow & G' & \xrightarrow{h|G'} & D_4 \longrightarrow 1 \end{array}$$

$Z_3 = T_{12}/D_4$  acts on the crystal  $\mathbb{R}^3/G'$ . Since the point group  $G'$  of  $G'$  is isomorphic to  $D_4$ , inspection of Table 4 tells us that  $\mathbb{R}^3/G'$  has three possible fibrations, corresponding to  $\mathbb{R}^3$  being fibred in the  $O_x$ ,  $O_y$  and  $O_z$  directions. The natural action of  $Z_3$  on  $\mathbb{R}^3/G'$  permutes these three fibrations cyclically and hence the three possibilities must be fibred-isomorphic to one another, giving exactly five possible crystals for  $\mathbb{R}^3/G'$ , each of which can be seen to admit a unique  $Z_3$ -action (up to conjugation). The five, with their corresponding groups  $G'$ , are:



(The last of these is the Seifert manifold  $Q_2$  of [Wa].)

The  $Z_3$ -actions on them yield  $\mathbb{R}^3/G$  as follows (everything has underlying space  $S^3$  and all singular points are of order 2, unless otherwise stated).



(6.1.1) to (6.1.3) and (6.1.5) are obvious. (6.1.4) can be seen as follows: from §4, we know that

$$\bar{Q}_2 \cong \frac{1}{2} \langle \text{figure} \rangle \frac{1}{2} \quad \text{is double covered by} \quad \langle \text{figure} \rangle \cong Q_2$$

i.e. there is a fibred involution on  $Q_2$  with quotient  $\bar{Q}_2$ . The  $\mathbb{Z}_3$ -action on  $\bar{Q}_2$  which is given in (6.1.5) must lift to a  $\mathbb{Z}_3$ -action on  $Q_2$  whose quotient is a 2-fold cover of  $\bar{Q}_2/\mathbb{Z}_3$  i.e. of:

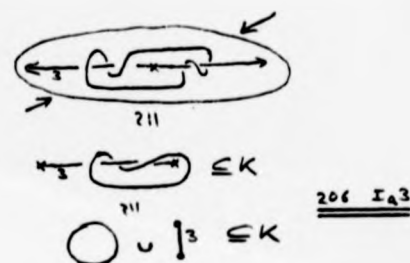
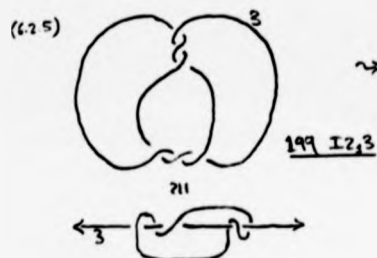
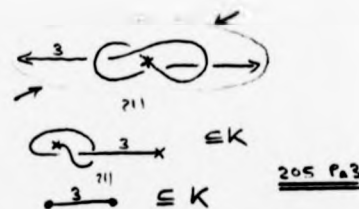
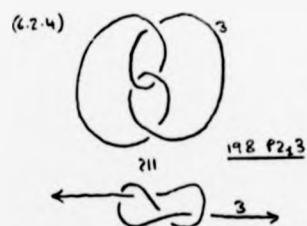
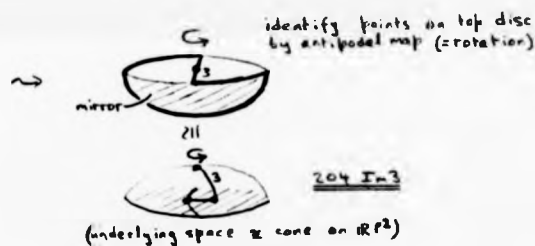
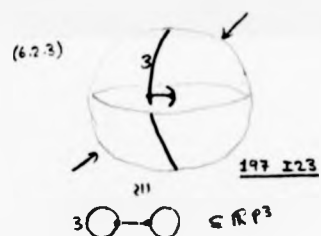
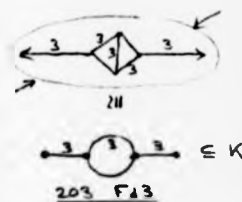
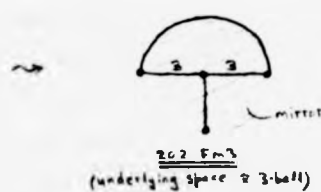
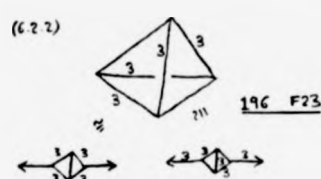
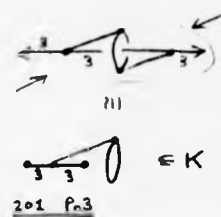
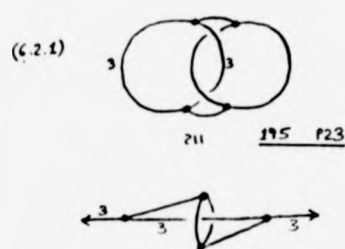


But the only possibility is the covering formed by unwrapping along the order 2 singular set, giving:

$$\langle \text{figure} \rangle^3 \cong Q_2/\mathbb{Z}_3.$$

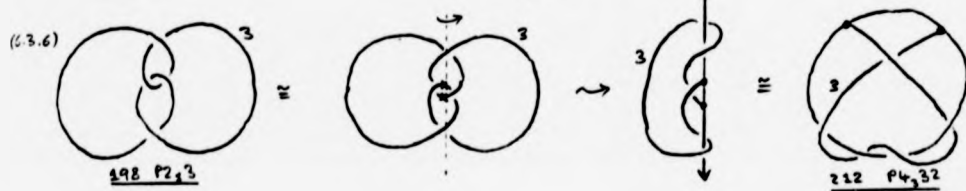
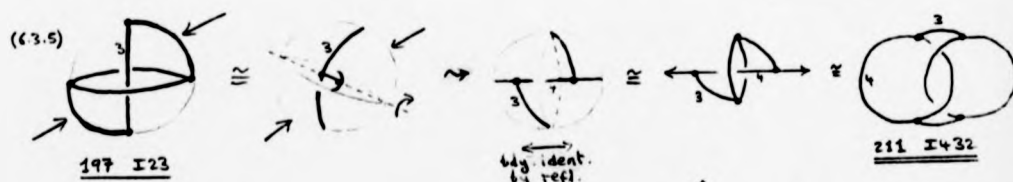
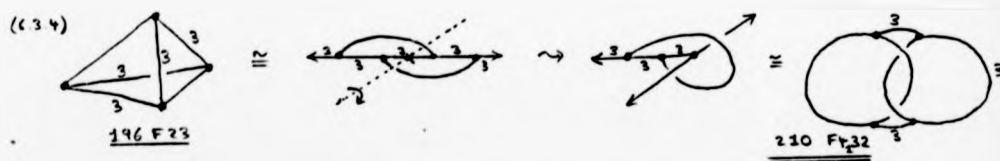
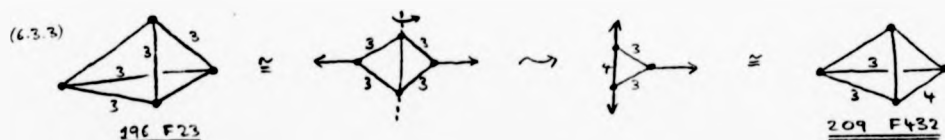
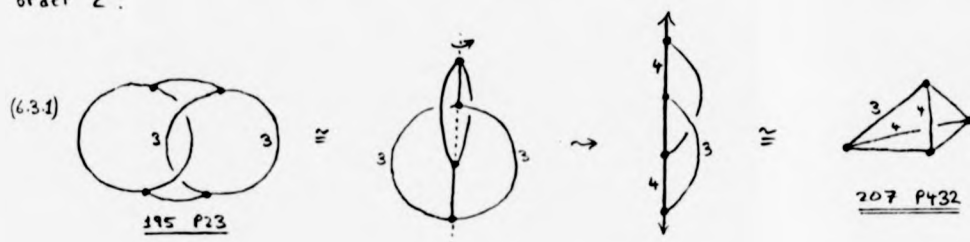
A further argument (see [B + S:Y-53]) shows why this is the *only* isometric  $\mathbb{Z}_3$ -action up to (isometric) conjugation. We omit it here.

(6.2)  $G_\infty \cong \hat{T}_{24}$ : Groups  $G$  having  $G_\infty \cong \hat{T}_{24}$  have crystals which are the quotient by an orientation-reversing involution of one of the crystals which were derived in (6.1.1) to (6.1.5). In what follows,  $K$  is the suspension on  $\mathbb{RP}^2$ : it is therefore *not* a manifold.

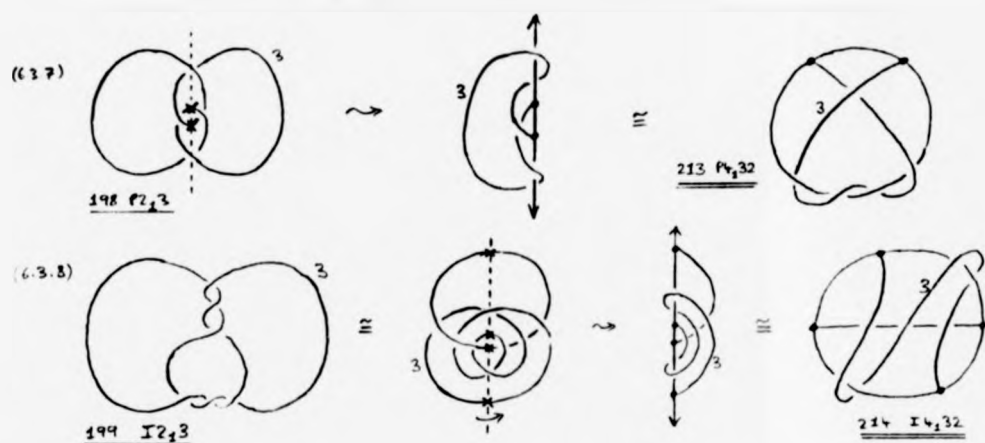




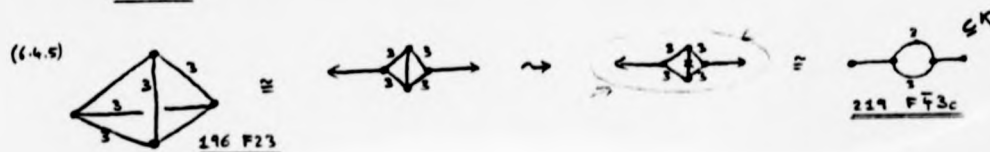
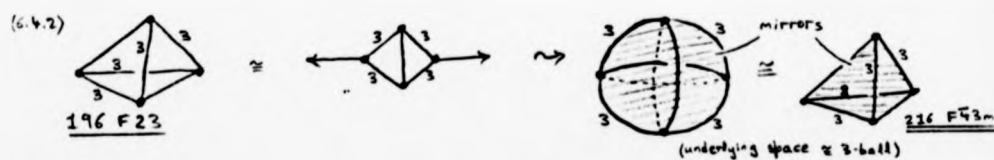
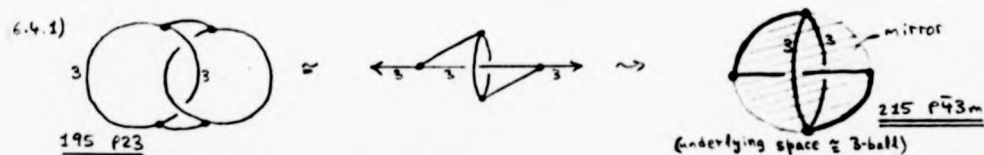
(6.3)  $G_{\infty} \cong O_{24}$  : These are the crystals which are the quotients of those in (6.1) (i.e.  $G_{\infty} \cong T_{32}$ ) by an orientation-preserving map of order 2.

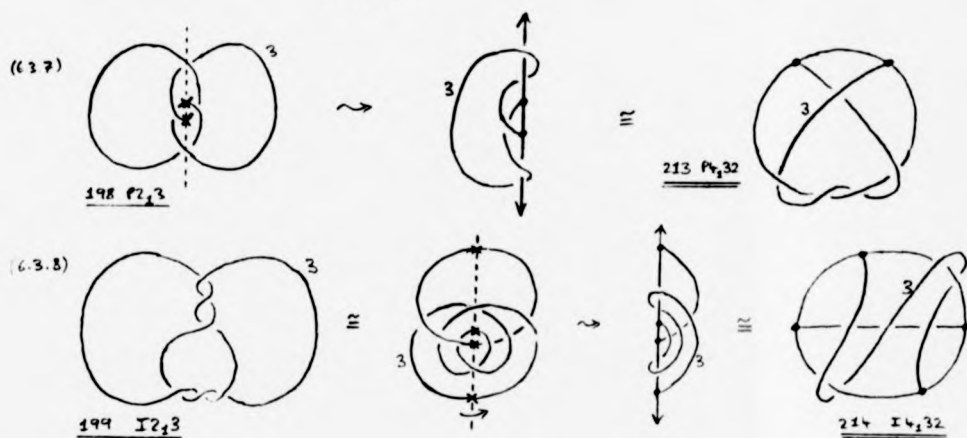


This and the next are an amphicheiral pair i.e. the two (orientable) 3-crystals are isomorphic by an orientation-reversing map.

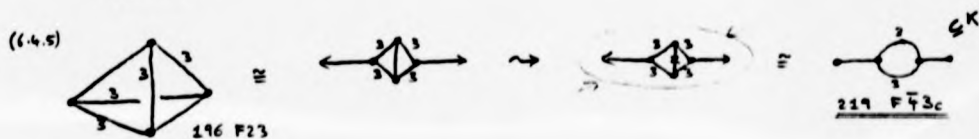
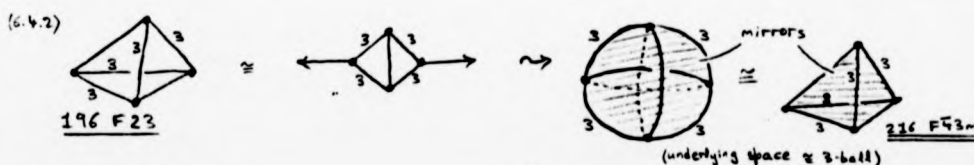
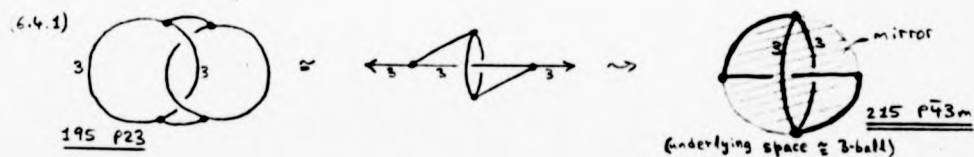


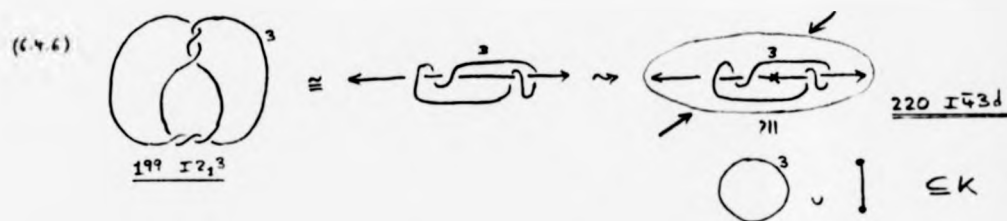
(6.4)  $G_{\alpha} \cong \bar{T}_{24}$ : These are all the quotients by orientation-reversing order 2 maps of the crystals in (6.1) which were not dealt with in (6.2) ( $G_{\alpha} \cong \hat{T}_{24}$ ).



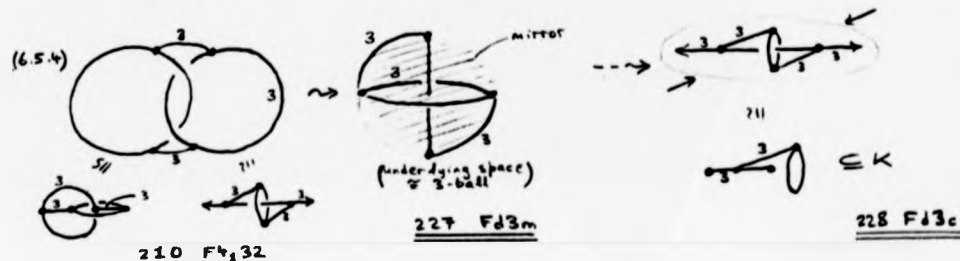
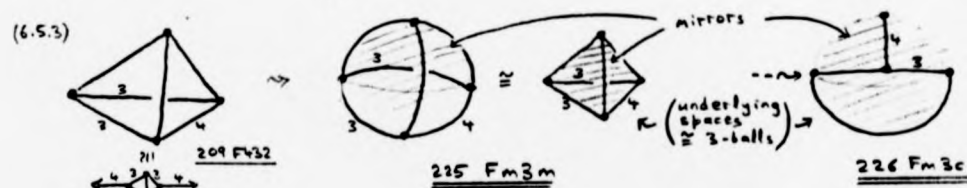
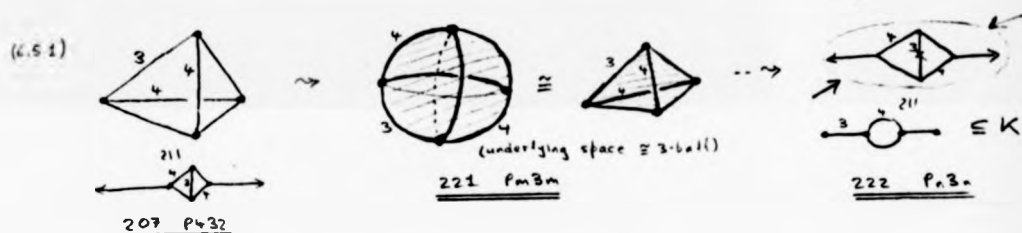


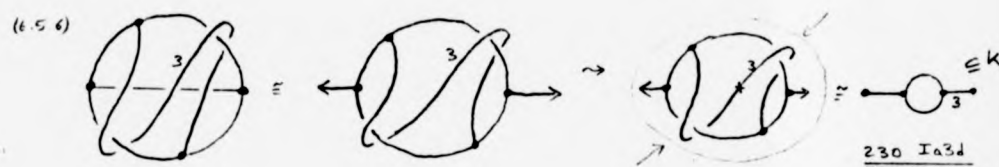
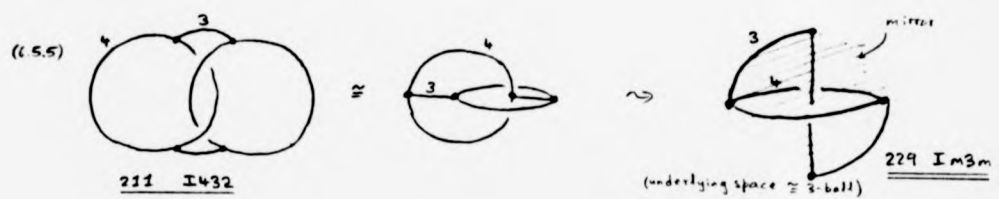
(6.4)  $G_{\infty} \cong \bar{T}_{24}$ : These are all the quotients by orientation-reversing order 2 maps of the crystals in (6.1) which were not dealt with in (6.2) ( $G_{\infty} \cong \hat{T}_{24}$ ).





(6.5)  $G \cong \bar{O}_{48}$  Finally, we have the crystals which are formed as quotients of the (orientable) crystals obtained in (6.3) (i.e. those with  $G_0 \cong O_{24}$ ).





§7. Examples.

Having produced at great length all the machinery of §§2, 4 and 5, we must now justify its existence. Our aim throughout has been to extend the topological classification of crystallographic groups in  $[B + S]$  to a classification of the black and white groups, by constructing fibred double covers of the  $S^1$ -fibred 3-crystals in Table 4, and by other techniques for the crystals which admit no  $S^1$ -fibration, listed in §6.

The conventional classification (see  $[B + C]$ , for instance) forewarns us that there are 1191 black and white groups (i.e. Type III and Type IV Shubnikov groups - see (3.1.3) and (3.1.4)), and so it is obviously not desirable to list here all the double coverings which correspond to them. Instead we shall illustrate their derivation by some examples, an understanding of which should enable readers to construct others for themselves.

The process of building double covers for all the necessary  $S^1$ -fibred 3-crystals has been described completely in §4 and §5, but a brief reprise here may be helpful. Starting with  $Q \rightarrow B$  from Table 4, where  $B$  is one of eleven 2-crystals (not the annulus, Möbius band or Klein bottle - see (4.0)), we first construct the "D-covers", i.e. those for which the base  $B$  is "unwrapped". Double covers  $B'$  of  $B$  are read off from Table 1, the local structure of  $Q' \rightarrow B'$  lifts from that of  $Q \rightarrow B$  (using Table 3.1 when the generic fibre of  $Q \rightarrow B$  is  $S^1$  and Table 3.2 when it is  $m\mathbb{I}$ ) and any other aspects of the structure of  $Q' \rightarrow B'$ , not forced by the local structure, are calculated using (4.5) to (4.8) or (4.10) to

(4.11), again depending on the generic fibre of  $Q \rightarrow B$ .

The "S-covers" of  $Q \rightarrow B$  are dealt with in §5. These fibre over the same base  $B$  i.e. they are formed by "unwrapping the fibres". For a given  $Q \rightarrow B$  such covers may not exist at all, since some local fibre types do not possess S-covers (5.1), (5.2). However, when all the local fibre types of  $Q \rightarrow B$  do admit them we construct  $Q' \rightarrow B$  in a piecemeal fashion using Table 3.3 when the generic fibre of  $Q \rightarrow B$  is  $S^1$  and Tables 3.4 and 3.5 when it is  $mI$ . There may be more than one choice of covering for each local fibre type and we must take every possible combination when producing a list (with the proviso that we do so in a consistent fashion i.e.  $Q' \rightarrow B$  must "make sense" so, in particular, fibrations from Tables 3.4 and 3.5 cannot be mixed). Again, having generated the local structure, if this completely determines  $Q' \rightarrow B$  then we stop; otherwise, the remaining invariants are found by using (5.1.1) to (5.1.7) when  $Q' \rightarrow B$  comes from Table 3.3, (5.2.1) to (5.2.3) when it comes from Table 3.5, and (5.3.1) to (5.3.4), when it comes from Table 3.4.

Those fibrations in Table 4 which are not unique (in the sense of Definition 3.4) are accompanied by the words "many", "twice", or "three times". If our initial  $Q \rightarrow B$  is unique, and all the fibred double covers in our list are inequivalent (so for any two which happen to be fibred-isomorphic to one another, this isomorphism does not project to a fibred automorphism of  $Q \rightarrow B$ ) then, by Proposition (3.5), our list corresponds

exactly to the black and white groups  $(G, G')$ , where  $Q = \mathbb{R}^3/G$  and  $Q' = \mathbb{R}^3/G'$ . Here are some examples, with  $G$  denoted by its number and symbol in Table 4. Each double cover of  $Q \rightarrow B$  has been labelled with the conventional notation for black and white groups, as found in  $[B + C]$  for instance, and those double covers which are not determined completely by their local structure are further annotated with the source(s) of the additional information. For the first example the covering maps on the base  $B'$  of the base-unwrapped cases have been indicated; for the others, we omit them.

(7.1.1) Example:

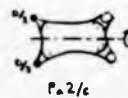
13.  $G \cong P2/b$  ;  $Q \rightarrow B$  is :



Unwrap  
Base  
(Table 3.1)



$P2'/c'$



$P_2 2/c$

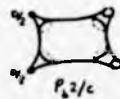


$P2'/c'$



$P2'/c$  (4.7.3) (4.8.9)

Fix Base  
(Table 3.3)



$P_2 2/c$



$C_2 2/c$



$P_2 2_1/c$

We immediately have an example, viz:

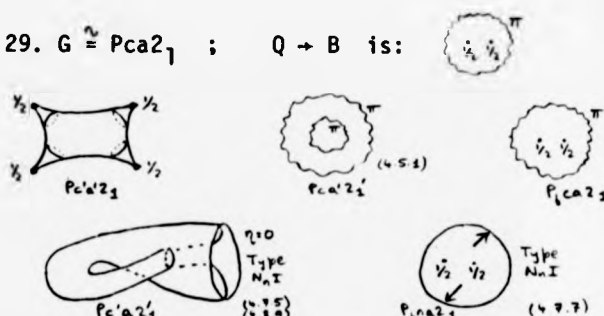


of a 3-crystal which covers  $Q \rightarrow B$  in two distinct ways, corresponding to different black and white groups. This is precisely what leads to the requirement in the definition of equivalence of fibred double covers (3.3.1) that the fibred isomorphism between them must project to a fibred automorphism of  $Q \rightarrow B$ , which it clearly cannot do in this case. //



(7.1.2) Example: 29.  $G \cong Pca2_1$  ;  $Q \rightarrow B$  is:

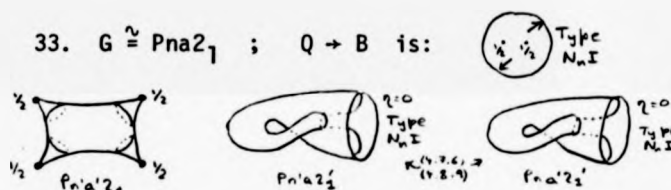
Unwrap  
Base  
(Table 3.1)



We cannot take covers over a fixed base because  $\pi$  and  $\frac{1}{2}$  fibrations do not possess such covers. //

(7.1.3) Example: 33.  $G \cong Pna2_1$  ;  $Q \rightarrow B$  is:

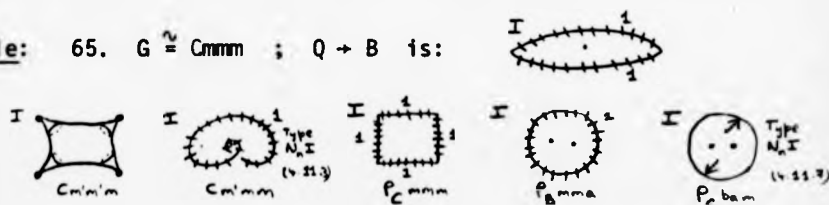
Unwrap  
Base  
(Table 3.1)



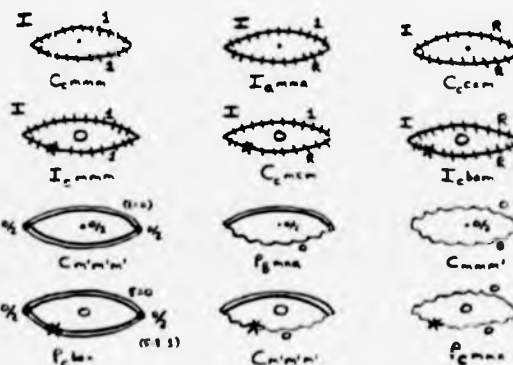
Again there can be no covers over a fixed base. The reason that  $Q \rightarrow B$  is twice covered by the same object in distinct ways was explained in (4.4.1). //

(7.1.4) Example: 65.  $G \cong Cmmm$  ;  $Q \rightarrow B$  is:

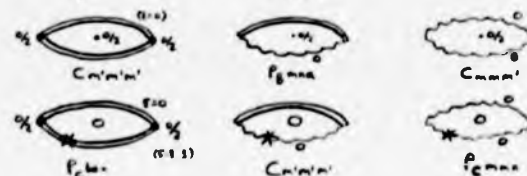
Unwrap  
Base  
(Table 3.2)



Fix  
Base  
(Table 3.5)




Fix  
Base  
(Table 3.4)




This is our first example for which  $Q \rightarrow B$  has generic fibre  $mI$  as opposed to  $S^1$ . As we can see, it is essentially no different, except that there are two distinct types of cover over a fixed base: those with generic fibre  $mI$ , and those with generic fibre  $S^1$ . //

All the examples so far have involved taking double covers of crystals which have a unique fibration. Consequently, once we have a complete list of inequivalent fibred double covers we are done; each cover corresponds to a distinct black and white group. We remarked in (3.6.1) that some groups preserve more than one direction in  $R^3$  and, for 14 of them, there is no direction which yields a unique fibration. As promised there, the first two of these are dealt with individually:


(7.2.1) Example: 1.  $G \cong P^1$ ;  $Q \rightarrow B$  is:   $T_{3\pi} \cdot$

This orbifold is just  $S^1 \times S^1 \times S^1$ , since  $G$  is generated by any three translations in linearly-independent directions. It is easy to see that there is only one black and white group  $(G, G')$  (up to equivalence by affine homeomorphisms of  $R^3$ );  $G'$  is just a copy of  $G$  lying as any index two subgroup, and corresponds to every double cover. //

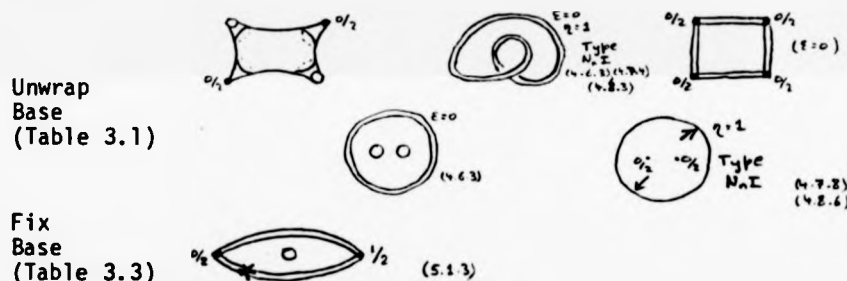
(7.2.2) Example: 2.  $G \cong P^1$ ;  $Q \rightarrow B$  is: 

Here  $G$  has exactly the same generators as the above example, together with an involution through the origin. Thus there are only two possibilities for  $(G, G')$ : either  $G'$  is a copy of  $P^1$ , lying in  $P^1$  as an index two subgroup; or  $G'$  is a copy of  $P^1$ . The double covers to which each corresponds are obvious. //

This leaves 12 more "serious" examples of groups  $G$  whose orbifolds have no unique fibration. Arguing as in (3.6.3), we will give two examples of how to determine the black and white pairs  $(G, G')$  up to equivalence; the other 10 follow just as easily.

(7.2.3) Example: 48.  $G \cong Pnnn$ ;  $Q \rightarrow B$  is: 

To start off, we proceed just as before and produce a list of all the inequivalent fibred double covers:



Because the fibration of  $Q \rightarrow B$  is not unique, we cannot use Proposition 3.5 to assert that each of these covers corresponds to a distinct black and white group. As we noted in (3.6.3), since  $G$  preserves three mutually perpendicular directions in  $R^3$  (all leading to the same fibration), if  $Q' = R^3/G'$  is a double cover of  $Q$  which itself admits more than one distinct fibration then the principal one for  $Q'$  (i.e. the first in Table 4) and at least one other fibration for  $Q'$  (or both if these exist and  $G'$  is orthorhombic) must show up in the list of double covers of  $Q \rightarrow B$ . This enables us to "pair off" covers from our list which correspond to the same black and white group  $(G, G')$ , by using Table 4 to



It remains for us to treat those crystals which are the quotients of groups in the cubic class. We met these in §6 and saw that although such a group  $G$  does not preserve any direction in  $\mathbb{R}^3$  it still makes sense for us to look at  $Q = \mathbb{R}^3/G$ , despite the fact that it no longer admits an  $S^1$ -fibration. We know from §6 that  $Q$  will always have underlying space  $|Q|$  homeomorphic to  $S^3$ ; a 3-ball;  $\mathbb{RP}^3$ ;  $K$  (the suspension on  $\mathbb{RP}^2$ ); or the cone on  $\mathbb{RP}^2$ . Since the first two have trivial fundamental group and the other three have  $\pi_1(|Q|) = \mathbb{Z}_2$ , the following is of great use [Du 1:Prop.1.1]:

(7.3.0) Proposition: If  $Q$  is a good  $n$ -orbifold with underlying space  $|Q|$  and  $N$  is the (normal) subgroup of  $\pi_1(Q)$  generated by covering translations fixing points, then

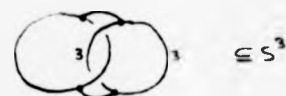
$$\pi_1(|Q|) \cong \pi_1(Q)/N.$$

□

Thus, when  $|Q| \cong S^3$  or a 3-ball we have  $\pi_1(Q) \cong N$  i.e. the generators of  $\pi_1(Q) (\cong G, \text{ our crystallographic group})$  all contribute to  $\Sigma_Q$ , the singular set. In such cases, if we wish to form a double cover  $Q'$  of  $Q$  we can do so only by unwrapping the singular set in some suitable way.

When  $|Q|$  is one of the others we have  $\pi_1(|Q|) \cong \pi_1(Q)/\mathbb{Z}_2$ . This gives us two options: either we can unwrap the singular set or double cover the underlying space.

(7.3.1) Example: 195.  $G \cong P23$  ;  $Q$  is



We can re-draw  $Q$  with the singular set lying as follows:



From this, and the fact that  $\pi_1(|Q|) = \pi_1(S^3) = \{id\}$ , it follows that there is only one way to double cover  $Q$  and that is by "unwrapping" along the horizontal  $\mathbb{Z}_2$ -axis. Trying to do the same along any other part of the singular set is impossible, since it would involve unfolding at points with local group  $\mathbb{Z}_3$ . Thus the only double cover is:

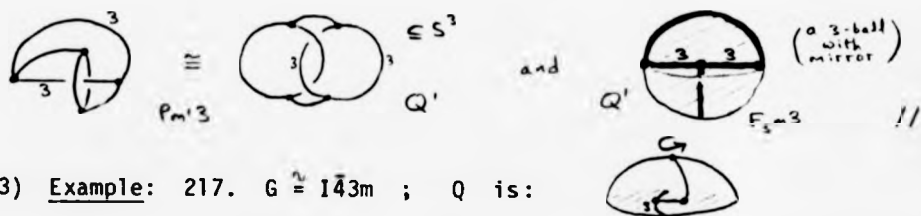


We recognize this as the quotient of group number 196 ( $F23$ ) and so we know that there is only one black andwhite group  $(G, G')$  for which  $G \cong P23$ ; it has  $G' \cong F23$  and is denoted by  $F_5 23$  in [B + C]. //

(7.3.2) Example: 200.  $G \cong Pm3$  ;  $Q$  is:



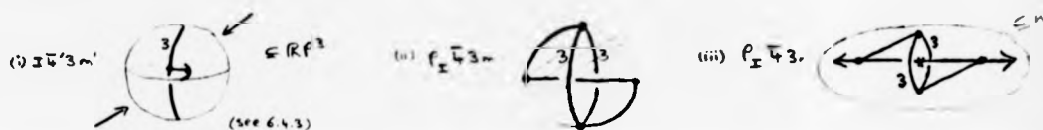
Again we have trivial  $\pi_1(|Q|)$  and so we must unfold the singular set to obtain  $Q'$ ; note that here  $\Sigma_Q$  consists not only of 1-dimensional components but also has a 2-dimensional component, namely the mirror which lies as  $\partial|Q|$ . This gives us two possibilities: either double along the mirror or unwrap along the circle of singular points with local group  $\mathbb{Z}_2$  (vertical in the picture). Anything else fails to make sense or involves unfolding along points which are singular of order 3, yielding an unwanted 3-fold cover. We get:



(7.3.3) Example: 217.  $G \approx I\bar{4}3m$  ;  $Q$  is:

(The base is a mirrored  $\mathbb{R}P^2$  ; points on the upper hemisphere are identified antipodally, so the underlying space is the cone on  $\mathbb{R}P^2$  - see (6.4.3).)

We can cover the underlying space by  $\mathbb{R}P^3$ , a 3-ball or  $K$  ; respectively, these give:



Unwrapping along the singular set is not possible in this case; the non-manifold point at the centre of the hemisphere (and lying on a set of  $\mathbb{Z}_2$ -points) prohibits this. //

(7.4) Those readers who wish to should now be able to determine any other covers, and the corresponding black and white groups, for themselves. We have therefore completed the task that we set ourselves: to provide a topological classification of the black and white groups. It is not just a copy of the classical scheme of things, either. Whereas for a given group  $G$  we are able to determine all the inequivalent pairs  $(G, G')$ , the way that black and white groups are usually presented *does this only* for the Type III Shubnikov groups i.e. those for which  $G'$  differs from  $G$  in its point group. The Type IV groups, those for which  $G'$  differs from  $G$  in its lattice, are referred not to  $G$  but to  $G'$ . This is evident in the conventional notation. For example, in (7.3.3) we have  $G = I\bar{4}3m$  and three pairs  $(G, G')$  with  $G'$  respectively being  $I23$ ,  $P\bar{4}3m$  and  $P\bar{4}3n$ . The corresponding black and white groups are labelled  $I\bar{4}'3m'$ ,

$P_1\bar{4}3m$  and  $P_1\bar{4}3n$ . The first is a Type III Shubnikov group, and the notation for it, with apostrophes on *the symbol for G* indicates this by showing that elements of the point group change colour. The other two are Type IV Shubnikov groups, and their notation shows, by subscripting *the symbol for G'* by 'I', that the colour change sends the lattice of  $G'$  (labelled 'P') to the lattice of  $G$  (labelled 'I').

This information is retrievable from our topological classification. Suppose we have a group  $G$ , and that we have found its corresponding 3-crystal  $Q$  in Table 4 (or §6 when  $\mathbb{R}^3/G$  cannot be fibred). We then make a list of its double covers and, perhaps with a little extra work, end up with a list of 3-crystals  $Q'$  which, as double covers of  $Q$ , correspond to black and white groups  $(G, G')$ . It is then possible, again by consulting Table 4 (or §6), to say which of these are Type III Shubnikov groups and which are Type IV. Precisely those  $Q'$  which are listed in the same geometric crystal class as  $Q$  (i.e. have the same  $G_\infty$  in Table 4 or §6) will correspond to Type IV groups  $(G, G')$ ; their point groups have not changed and so the colour change must be in the lattice. Those  $Q'$  which lie in a different geometric crystal class from  $Q$  (i.e. have different  $G_\infty$ ) must have "lost" some of their point group, and so represent Type III Shubnikov groups.

(7.5) Where can all of this lead us from here? Certainly these methods, or their equivalents, can be used to determine groups of more than two



colours (the *polychromatic* or *colour groups* - see [J + S] or [Schw.2]), both in three dimensions and in two, the latter giving us diagrams similar to those of Table 1. It remains to be seen whether such groups will have broad applications. More importantly, it is plausible that this topological approach will be more convenient than the purely group-theoretic alternative when attempting to understand what goes on in 4-dimensional crystallographic groups; with luck there will be a corresponding feedback to the infant study of 4-orbifolds.

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